LESSON 16

Read about this PROVISIONAL EDITION in the front matter to this book.
Check the NFB website periodically for updates to this lesson.

SYSTEMS OF EQUATIONS

16.1 Definition and Recognition: A *system of equations* is a collection of two or more equations which share variables. The student identifies the value of each variable by "solving the system." You can recognize a system by noticing the arrangement of equations on two or more lines. If the system consists of two equations, there will be two variables to solve (typically *x* and *y*); if the system consists of three equations, there will be three variables to solve (typically *x*, *y*, and *z*). The equations may or may not be joined by an enlarged grouping sign. Here is a print example of two equations, each with variables *x* and *y*.

$$4x - y = 10$$
$$2x = 12 - 3y$$

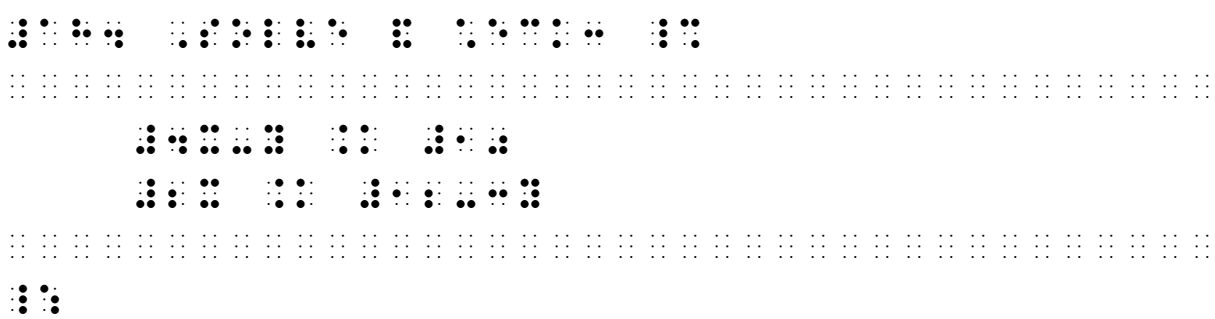
16.2 Transcription Rules for Systems of Equations: Systems of equations are brailled following rules of both spatial and nonspatial material, as follows.

— One blank line is left above and below the system, according to rules of the Nemeth Code for *spatial* material.
— The numeric indicator is (or is not) used according to rules of the Nemeth Code for *nonspatial* material.
— Alignment is maintained only if all terms and symbols are aligned in print.

Example 16.2-1 18. Solve and check:

$$4x - y = 10$$
$$2x = 12 - 3y$$

Print observations: This system has two variables (x, y) – two equations. The terms are not vertically aligned.



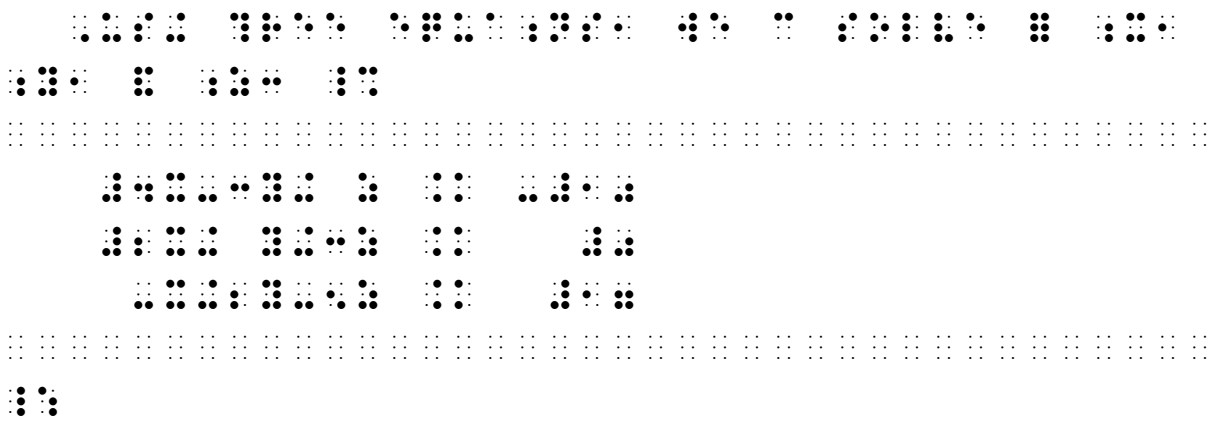
Braille observations: Each equation begins in the same cell because the terms are not vertically aligned in print. A blank line is inserted before and after the system. Numeric indicators are used as needed according to rules of the Nemeth Code for nonspatial material. Code switch indicators do not interfere with the arrangement. The Nemeth Code terminator is placed in cell 1 after a blank line following the spatial arrangement.

Note: The Nemeth Code calls such an arrangement an "ununified system of equations" because the equations are not joined by a grouping symbol.

Example 16.2-2 Using three equations, we can solve for x, y, and z:

$$\begin{aligned} 4x - 3y + z &= -10 \\ 2x + y + 3z &= 0 \\ -x + 2y - 5z &= 17 \end{aligned}$$

Print observation: The terms on both sides of the equals signs are vertically aligned.



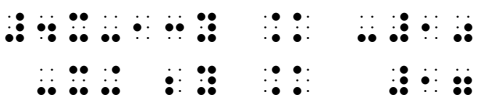
Braille observations: A blank line is inserted before and after the system. Spaces are inserted in order to maintain alignment as printed. Numeric indicators used as needed according to rules of the Nemeth Code for nonspatial material.

The BANA Nemeth Code Technical Committee is discussing details regarding the situation encountered below. A provisional solution is provided.

When numerical coefficients with a different number of digits need to line up by place value in order to follow print alignment, a numeric indicator is not inserted in the space.

Example 16.2-3

$$\begin{aligned} 4x - 13y &= -10 \\ -x + 2y &= 17 \end{aligned}$$



A space is inserted before the single digit "2" to achieve alignment with the "13" above it.

PRACTICE 16A

1. Solve and check:

$$\begin{aligned} 2x + 3y &= 2 \\ 8x - 4z &= 3 \\ 3y - 8z &= -1 \end{aligned}$$

2. Solve:

$$\begin{aligned} 2x - 5y + 6z &= 11 \\ 3x - 2y + 3z &= 9 \\ 2x + 4y - 9z &= -3 \end{aligned}$$

16.3 Unified System of Equations and Enlarged Grouping Signs: The Nemeth Code calls an arrangement a "unified system of equations" when the equations are joined by an enlarged grouping symbol. Here is a print example, using an enlarged left brace.

$$\begin{cases} x = y \\ 5x - y = 4 \end{cases}$$

Enlarged Signs of Grouping

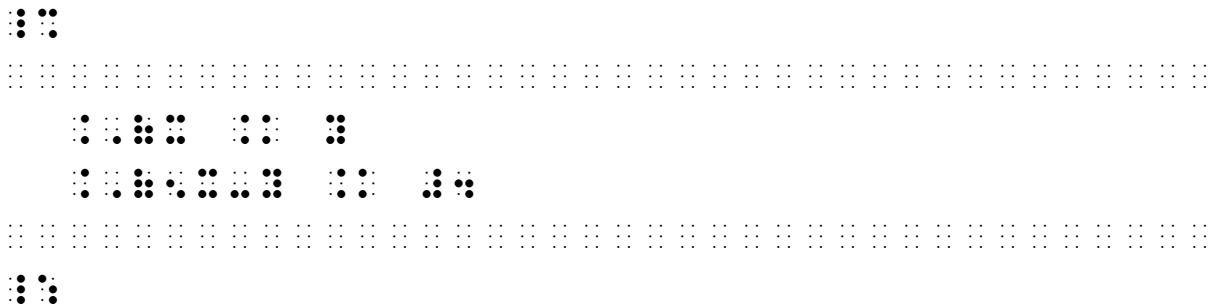
16.4 Transcription Rules for Enlarged Signs of Grouping: In braille, enlarged signs of grouping are used on each line of the unified system of equations and are vertically aligned. The opening and the closing enlarged grouping symbol are placed in the cells next to the items which extend furthest left and furthest right, with no intervening space. If only the opening or closing sign of grouping is shown in print, only that sign is shown in braille.

16.4.1 Enlarged Left Brace: The enlarged left brace curves and points to the left in print. Notice that the enlarged braille symbol is formed by inserting a dot 6 before the ⠸ symbol of the normal brace symbol.

Enlarged Left Brace	{	⠈⠸	(on two or more lines)

Example 16.4-1 $\begin{cases} x = y \\ 5x - y = 4 \end{cases}$

Print observations: An enlarged left brace groups the unified system of equations. Terms are not vertically aligned.



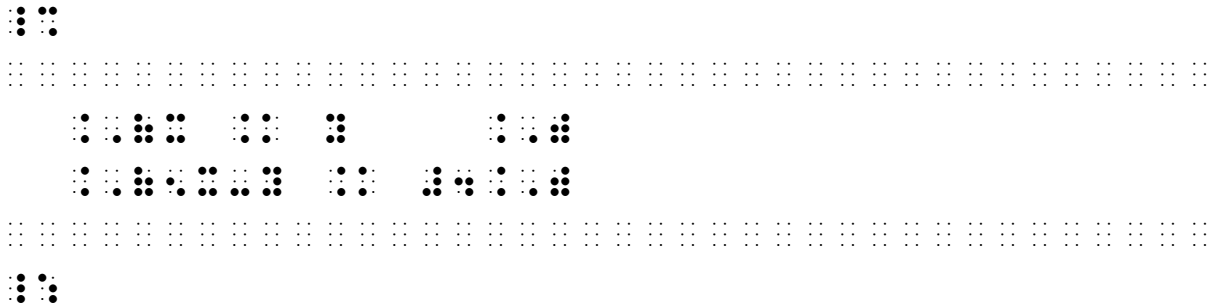
Braille observations: Each enlarged grouping symbol begins in the same cell. When terms are not aligned in print, each equation begins in the cell following the left enlarged brace symbol. Numeric indicators are used as needed according to rules of the Nemeth Code for nonspatial material.

16.4.2 Enlarged Right Brace: The enlarged right brace curves and points to the right in print. Notice that the enlarged braille symbol is formed by inserting a dot 6 before the ⠸ symbol of the normal brace symbol.

Enlarged Right Brace	}	⠸	(on two or more lines)

The same system is shown below, with an enlarged right brace added.

Example 16.4-2 $\begin{cases} x = y \\ 5x - y = 4 \end{cases}$

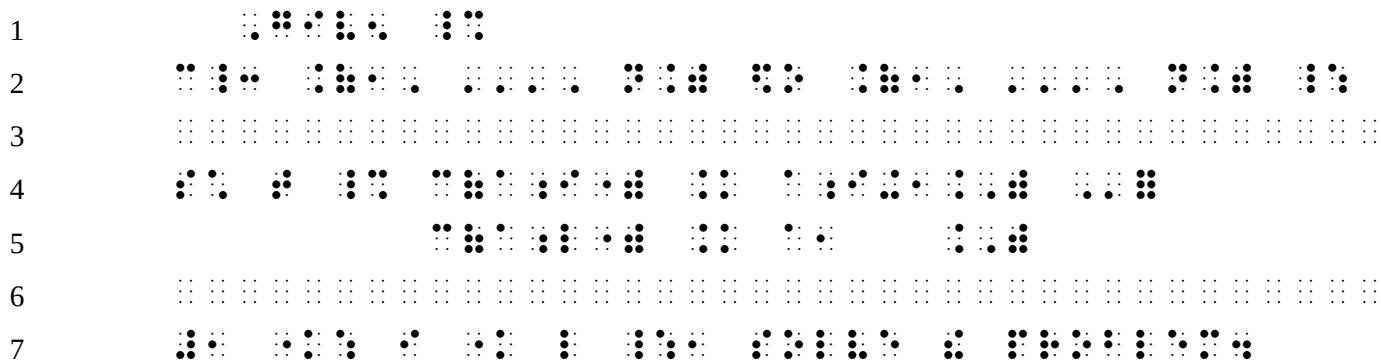


Closing grouping symbols are vertically aligned, starting in the cell next to the longest equation.

16.5 Embedded Vertical Groupings: Enlarged grouping signs are used in other structures as well, not only with systems of equations. If the grouping is embedded within narrative, the required blank lines before and after the arrangement become part of the paragraph. The surrounding text is placed on the top line of the arrangement only – before and/or after the math as it fits in context of the narrative. Text continues in the runover cell of the paragraph after the second required blank line. The opening Nemeth Code indicator and the Nemeth Code terminator are placed only on the top line of the embedded arrangement. The code switch indicators apply to the whole arrangement.

Example 16.5-1 Given $c: \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that

$$\left.\begin{array}{l} c(a_i) = a_{i+1} \\ c(a_l) = a_1 \end{array}\right\} \text{ for } 1 \le i < l \text{, solve the problem.}$$



Line 1: Paragraph begins in cell 3. (Lines 1-7 comprise the entire 3-1 paragraph.)
Lines 3 and 6: Blank line required before and after vertically-arranged grouping within the paragraph.
Line 4: Text is placed on top line of grouping. Opening switch placed on this line only.
Line 7:Paragraph continues following required blank line.

PRACTICE 16B

After solving for x and y, we determine that the system $\left\{\begin{array}{l} 2x - 3y = 17 \\ 3x + 2y = 6 \end{array}\right\}$ has the solution set $\{(4, -3)\}$. Name the solution set for this system:

$$\left\{\begin{array}{l} x + 2y = 6 \\ 2x - \ \ y = 7 \end{array}\right\}$$

16.6 Enlarged Parentheses: Notice that the enlarged braille parentheses are formed by inserting a dot 6 before the ⠷ or the ⠾ symbol of the normal parenthesis symbol.

Enlarged Parentheses			
Left	(	⠄⠷	(on two or more lines)
Right	)	⠄⠾	(on two or more lines)

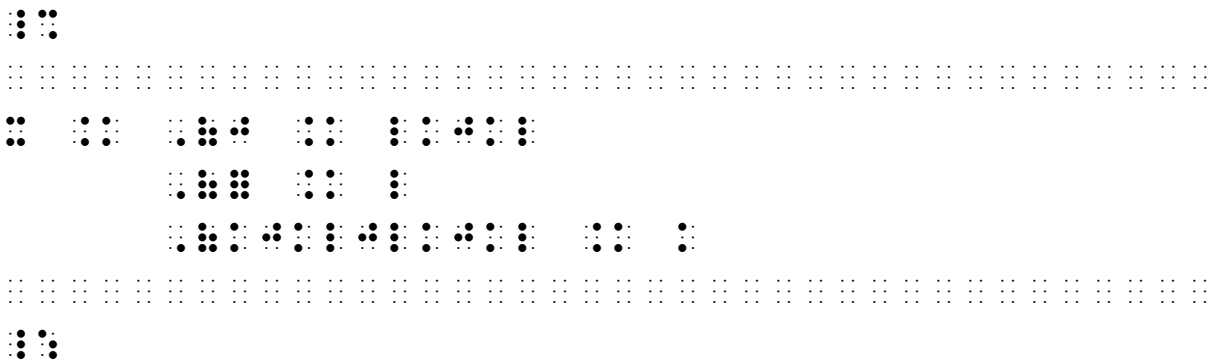
16.7 Placement of Symbols: Symbols which appear outside of the enlarged grouping symbol and which apply to the arrangement are placed on the top line of the arrangement, even if the items are centered in print. Also note the following details:

- The English letter indicator is (or is not) used according to rules of the Nemeth Code for nonspatial material.
- Alignment is maintained only if all terms and symbols are aligned in print.

Example 16.7-1

$$x = \left(\begin{array}{r} j = lkjkl \\ = l \\ kjkljlkjkl = k \end{array} \right.$$

Print observations: The anchor is centered with respect to the enlarged grouping symbol to its right. The three-line link is grouped with an enlarged left parenthesis. Equals signs in the grouped arrangement are aligned but terms are not.



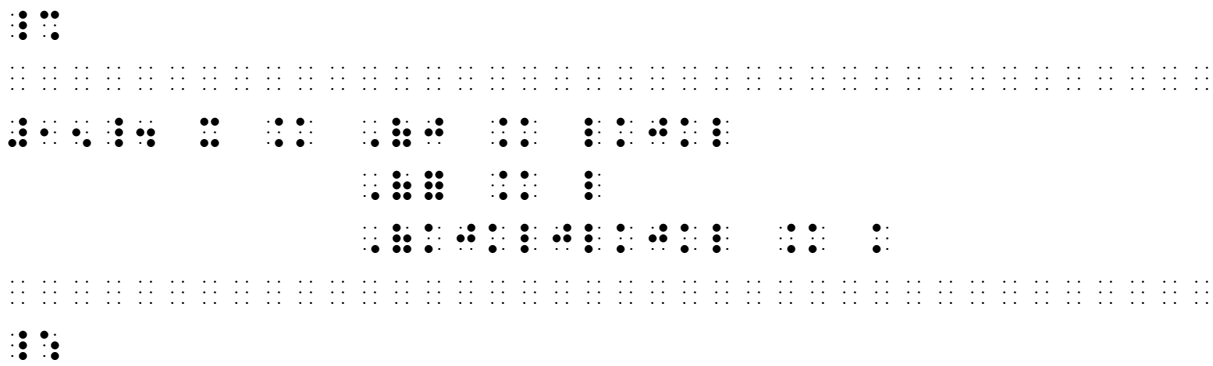
Braille observations: The anchor is aligned with the top line of the arrangement. Each enlarged grouping symbol begins in the same cell. The grouped items are left-adjusted because the terms are not aligned. A general omission symbol indicates the blank space before the middle equals sign. Code switch indicators do not interfere with the spatial arrangement.

16.8 Placement of Identifiers and Punctuation: Identifiers and punctuation which appear outside of the enlarged grouping symbol and which apply to the arrangement are placed on the top line of the arrangement, even if the items are centered in print. Here is the same example, now identified with an item number "15."

Example 16.8-1

15. $$x = \left(\begin{array}{r} j = lkjkl \\ = l \\ kjkljlkjkl = k \end{array} \right.$$

Print observation: The item number and the anchor are centered with respect to the associated enlarged grouping symbol to the right.

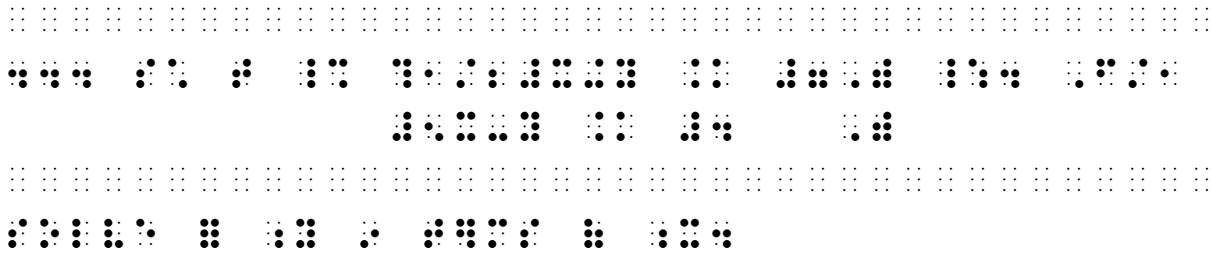


Braille observation: Item number and anchor are aligned with the top (first) line of the arrangement.

Example 16.8-2

... such that $\left. \begin{array}{r} \frac{1}{2}x + y = 7 \\ 5x - y = 4 \end{array} \right)$. First, solve for *y* in terms of *x*.

Print observations: A vertical arrangement is embedded within the narrative. It is grouped on the right with an enlarged right parenthesis. A period follows enlarged parenthesis, centered to the arrangement.



Braille observations: The narrative sentence ends with a displayed expression. The preceding text and the following period are brailled on the top line of the arrangement even though they are centered to the expression in print. A blank line is inserted before and after the lines with the embedded array.

16.9 Grouping Symbols Shown Taller in Print: Recall from **Lesson 3** that nested grouping symbols may be printed in different sizes in order to visually distinguish the nested symbols, but they are not brailled as "enlarged" symbols if they apply to only one line of characters.

⟫ $p\left(q\left(f\left(g\left(h\left(x\right)\right)\right)\right)\right)$ ⠏⠣⠟⠣⠋⠣⠛⠣⠓⠣⠭⠜⠜⠜

Similarly, grouping signs which enclose taller print constructions such as fractions, modified expressions, integrals, or binomial coefficients (**12.7.1**) are brailled as normal grouping symbols because they apply to only one line of braille characters.

⟫ $\left(\frac{3}{5}\right)^3 = \frac{27}{125} = 0.216$

⠣⠌⠼⠉⠱⠣⠌⠼⠑⠱⠣⠌⠼⠁⠃⠋⠱⠣⠌⠼⠁⠃⠋⠱⠣⠌⠼⠁⠁⠚⠱

⟫ $\left(\sum_k \binom{n}{k}\right)$ ⠣⠌⠎⠣⠌⠣⠌⠝⠜⠣⠌⠅⠜⠣⠌⠝⠜⠣⠌⠅⠜⠣⠌⠝⠜⠣⠌⠅⠜

PRACTICE 16C

A) Verify (YES or NO):

$$2\left(-\frac{4}{3}\right) - 1 \stackrel{?}{=} 4\left(-\frac{4}{3}\right) + 9$$

$$\left(-\frac{8}{3}\right) - 1 \stackrel{?}{=} \left(-\frac{16}{3}\right) + 9$$

$$-8 - 3 \stackrel{?}{=} -16 + 27$$

$$-11 = 11 . \quad \text{NO}$$

B) $\left(\begin{array}{l} \frac{x}{3} - \frac{y}{2} = 2 \\ 5x + 3y = 51 \end{array}\right) = (?, ?)$

The BANA Nemeth Code Technical Committee is discussing details regarding the treatment of remarks printed beside spatial arrangements. Correct transcriptions of the examples in this section will be confirmed after approval is obtained. For now, provisional solutions are provided.

16.10 Remarks Printed Next to Spatial Arrangements: Remarks which appear beside a spatial arrangement are brailled either beside the arrangement or after the blank line following the arrangement. Code switching is necessary if the remark contains narrative.

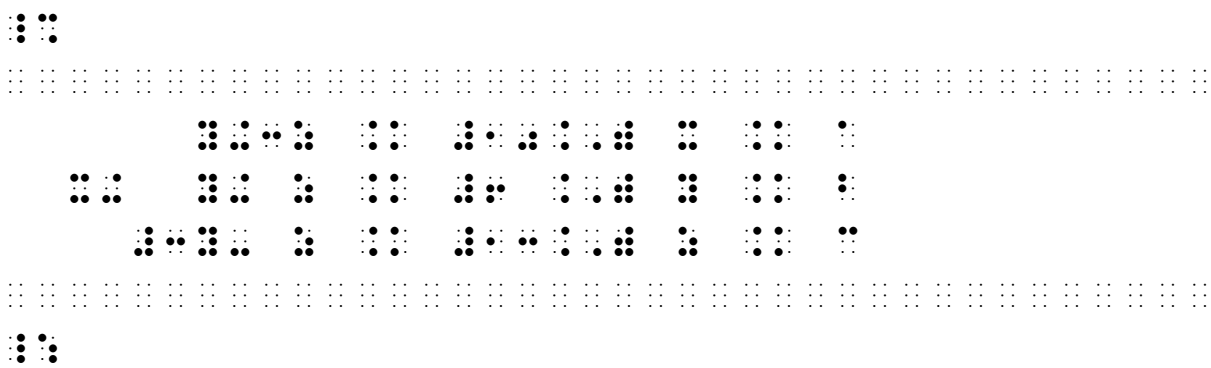
16.10.1 Remarks Brailled Beside Unified Expressions: Remarks printed to the right of an arrangement which is grouped with a right (closing) enlarged grouping sign can be brailled as printed as long as there is enough space remaining on the line for the remark. One blank cell is placed between the grouping symbol and the remark.

Example 16.10-1

$$\left.\begin{aligned} y+3z&=10\\ x+\ \ y+\ \ z&=6\\ 3y-\ \ z&=13 \end{aligned}\right\}\quad \begin{aligned} x&=a\\ y&=b\\ z&=c \end{aligned}$$

Print observations: Terms y and z on the left side of the equals signs are vertically aligned, although place values of the numbers to the right of the equals signs are not. Three individual remarks are printed to the right of the enlarged brace. The remarks are mathematical expressions.

PROVISIONAL TRANSCRIPTION



Braille observations: Spaces are inserted in the second equation in order to maintain alignment of factors and terms. Enlarged right braces are vertically aligned and begin in the cell next to the longest equation. One blank cell separates the closing grouping signs from each remark.

The next example shows a three-line unified system. A remark is printed to the right of the middle line of the system. In braille, the remark is moved to begin on the top line of the arrangement. If runovers are needed, they are indented two cells from the starting cell of the comment. Because this remark is not mathematical notation, a Nemeth Code terminator precedes the first word and the comments are transcribed in contracted braille.

Example 16.10-2

$$\left.\begin{aligned} y+3z&=10\\ x+\ y+\ z&=6\\ 3y-\ z&=13 \end{aligned}\right\}$$ *Solve by isolating each term.*

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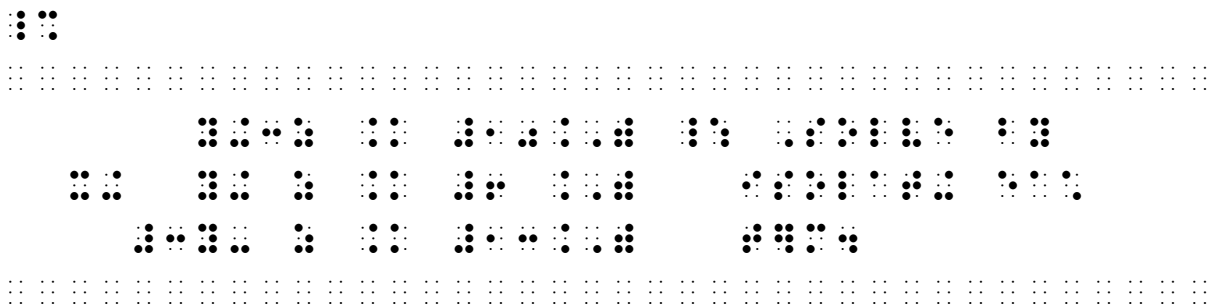

16.10.2 Remarks Brailled Below the Arrangement: The next example shows the same three-line unified system but the remark is longer. Because this remark takes more than the three available lines beside the system, it is placed after the arrangement in the runover cell for displayed material. If a Nemeth Code terminator is needed before the remark begins, the terminator is placed in cell 1 on the line following the required blank line.

Example 16.10-3

$$\left.\begin{aligned} y+3z&=10\\ x+\ y+\ z&=6\\ 3y-\ z&=13 \end{aligned}\right\}$$ *Solve this system by isolating each term: x, y, and z.*

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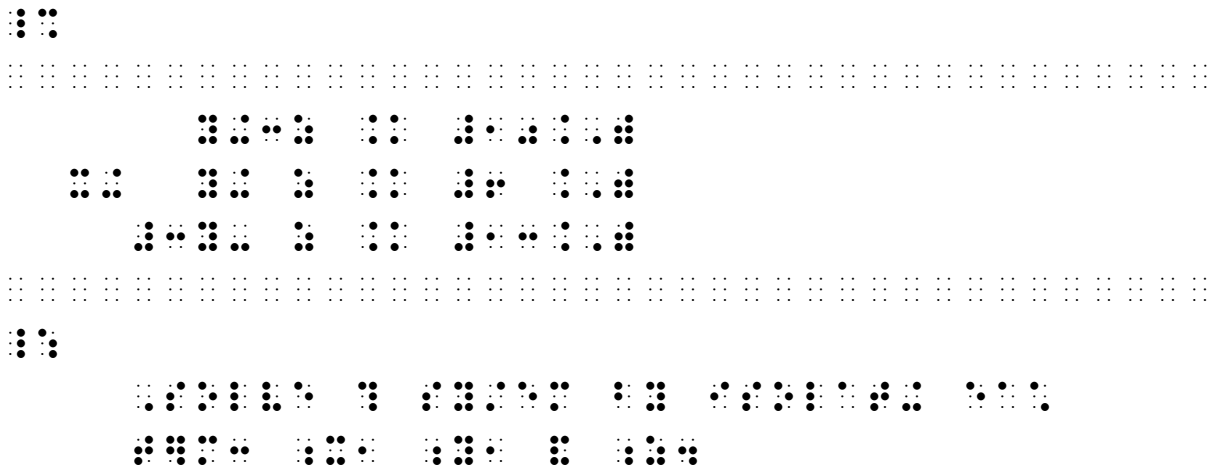

If there is no enlarged grouping sign to separate the math material from the remarks, place the remarks after the required blank line. The remark begins in the runover position.

Example 16.10-4 Thus, x and y must satisfy the constraints

$$\begin{cases} \frac{1}{4}x + \frac{1}{4}y \le 8 \\ \frac{1}{6}x + \frac{1}{3}y \le 8 \end{cases} \qquad x \ge 0, y \ge 0$$

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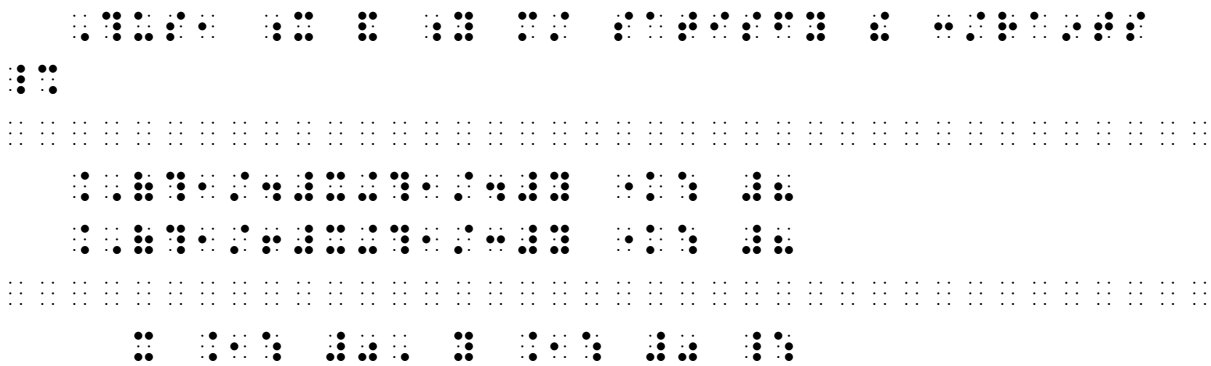
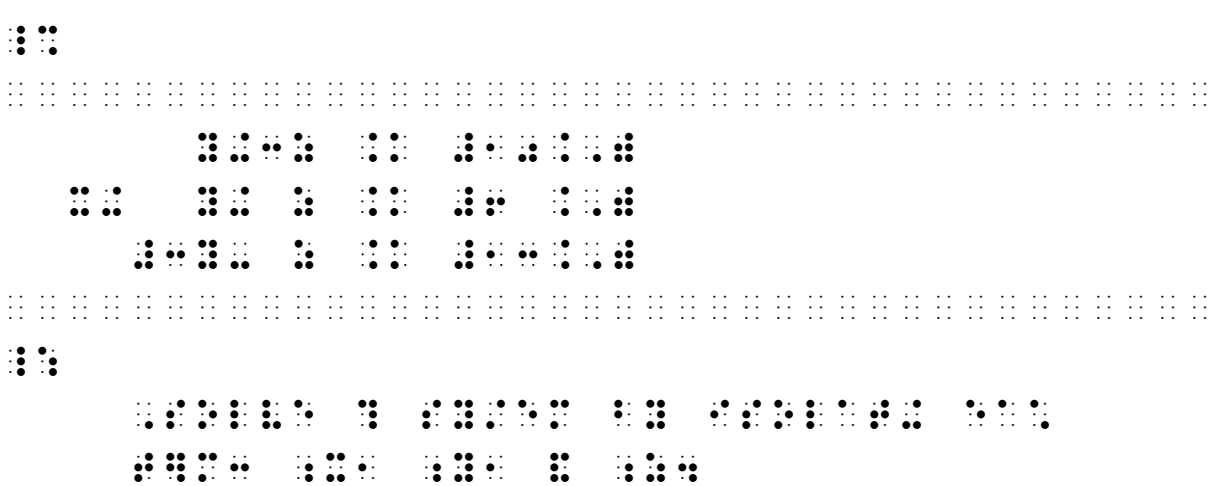


Example 16.10-5

$$\begin{aligned} y + 3z &= 10 \\ x + y + z &= 6 \\ 3y - z &= 13 \end{aligned}$$

Solve this system by isolating each term: x, y, and z.

PROVISIONAL TRANSCRIPTION



PRACTICE 16D

Find x and y in terms of a and b.

1. $\begin{cases} x + \ \ y = 0 \\ x + ay = 1 \end{cases} \qquad (a \neq 1)$

2. $\begin{cases} ax + by = 0 \\ x + \ \ y = 1 \end{cases} \qquad (a \neq b)$

3. $\begin{cases} ax + by = 0 \\ a^2x + b^2y = 1 \end{cases} \qquad (a \neq 0,\ b \neq 0,\ a \neq b)$

4. $\left.\begin{array}{l} a = \frac{x+y}{x-y} \\ \\ b = \frac{x-y}{x+y} \end{array}\right) \ -1 < x < 1, -1 < y < 1$

16.11 More Enlarged Signs of Grouping: In addition to the enlarged braces and parentheses, the Nemeth Code provides symbols for six other enlarged signs of grouping. Notice that each enlarged braille symbol is formed by inserting a dot 6 before the ⠸, ⠼, or ⠨ symbol of the normal symbol.

Enlarged Vertical Bar

Single |

Double ||

Enlarged Brackets

Left [**Right**]

Enlarged Angle Brackets

Left ⟨ **Right** ⟩

Enlarged Barred Brackets

Left **Right**

Enlarged Angle Braces

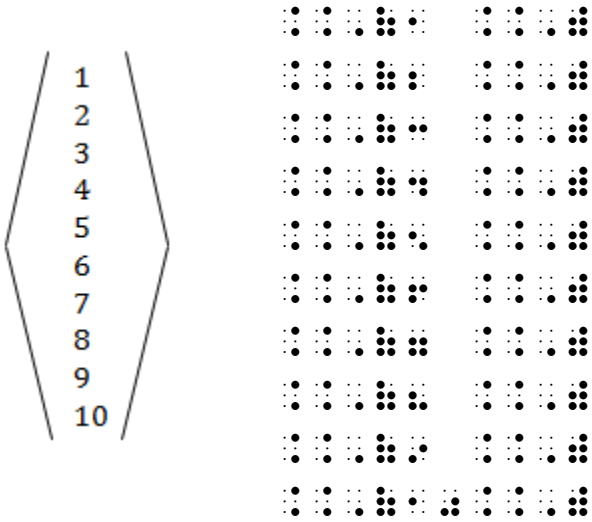
Left **Right**

Enlarged Half Brackets

Upper Left **Upper Right**

Lower Left **Lower Right**

Example 16.11-1



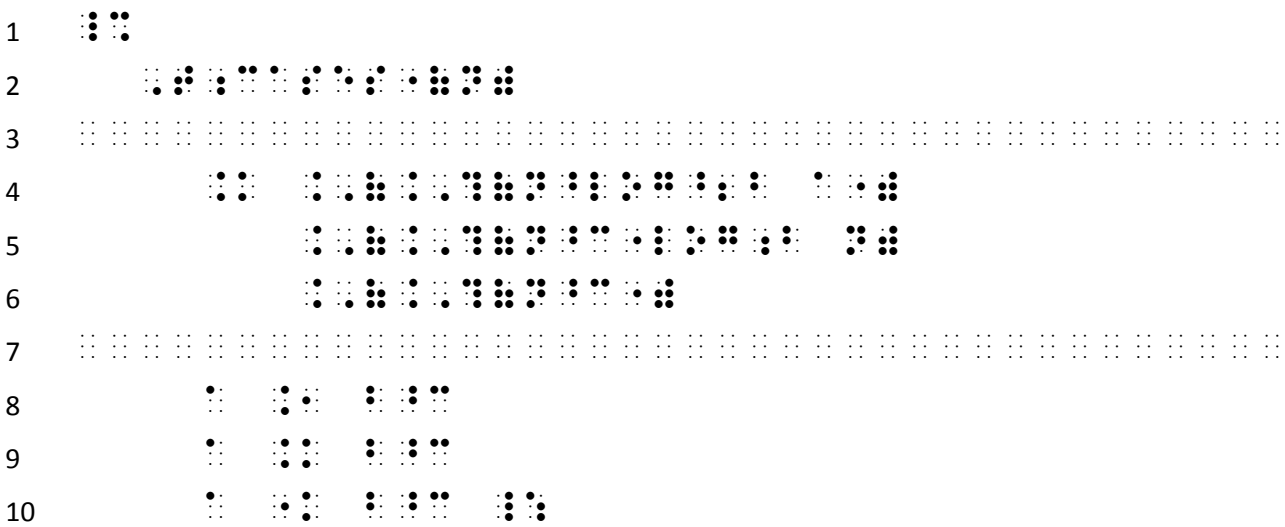
PRACTICE 16E

$$\langle x \rangle \quad \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \quad \left\langle \begin{matrix} u \\ v \\ w \\ x \\ y \\ z \end{matrix} \right\rangle$$

CHALLENGE PROBLEM #1

This problem shows an anchor followed by three lines of expressions grouped with an enlarged left brace. Conditions are printed to the right of each of the three lines. Each condition applies specifically to the expression to its left.

$$T_{cases}(n) = \begin{cases} \Theta\left(n^{\log_b a}\right) & a > b^c \\ \Theta(n^c \log_b n) & a = b^c \\ \Theta(n^c) & a < b^c \end{cases}$$



Line 2: Assuming this expression is displayed to narrative, it begins in cell 3. Because the "link" will not fit, the expression is divided according to the priority list of division of long expressions, before the comparison sign.

Lines 3 and 7: A blank line is inserted before and after the three-line grouped expression.

Line 4: The equals sign is in the runover position, cell 5.

Lines 8-10: Because there is no right brace, the remarks follow the arrangement. They are placed in the runover cell for displayed material.

Line 10: The remarks are not part of the spatial arrangement, so the Nemeth Code terminator is placed following the last math character.

DETERMINANTS AND MATRICES

16.12 Definition and Recognition: *Determinants* and *matrices* (singular: *matrix*) are arrangements of items in rows and columns which are enclosed between left and right grouping symbols. Items can be numbers, symbols, or mathematical expressions.

Here is a 2×2 ("two by two") determinant enclosed between enlarged vertical bars.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Here is a 2×3 ("two by three") matrix enclosed between enlarged brackets.

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$

Determinants and matrices may also be referred to as "arrays." An array can be composed of only one column or only one row. These will be studied at the end of this section.

16.13 Transcription Rules for Determinants and Matrices: Observe the following rules regarding the transcription of the two examples shown above.

16.13.1 Blank Lines: Determinants and matrices are spatial arrangements. Thus, a blank line is to be left above and below each array. If the arrangement begins at the top of a braille page, it may begin on line 1 provided no running head is in use; if the arrangement ends at the bottom of a braille page, it may end on line 25. In either case, the rightmost symbol of the arrangement must not fall within three cells of the page number. (See **10.31** in Lesson 10 for details regarding layout at the top or bottom of a page.)

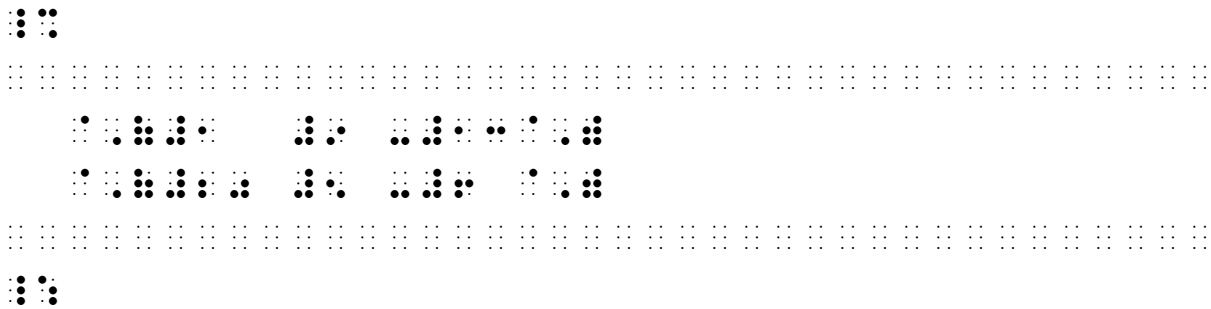
16.13.2 Grouping Symbols: For an array consisting of two or more rows, enlarged grouping symbols are used on each braille line. Grouping symbols are vertically aligned.

16.13.3 Placement of Items: Each entry is moved as far left as possible in its column. Consequently, each opening sign of grouping will be in direct contact with the first entry of each row in the array. Centering and other forms of alignment are not permitted. One column of blank cells is left between the columns of the arrangement. That is, one blank cell separates the widest entry in a column from the beginning of the next column. At least one closing sign of grouping must be in direct contact with an entry in the array. The widest entry in the last column determines the placement of the right (closing) enlarged grouping signs.

16.13.4 Numeric and Letter Indicators: The numeric indicator is used with numeric entries in an array, even when such entries are in direct contact with an opening grouping sign.

Example 16.13-1 $\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$

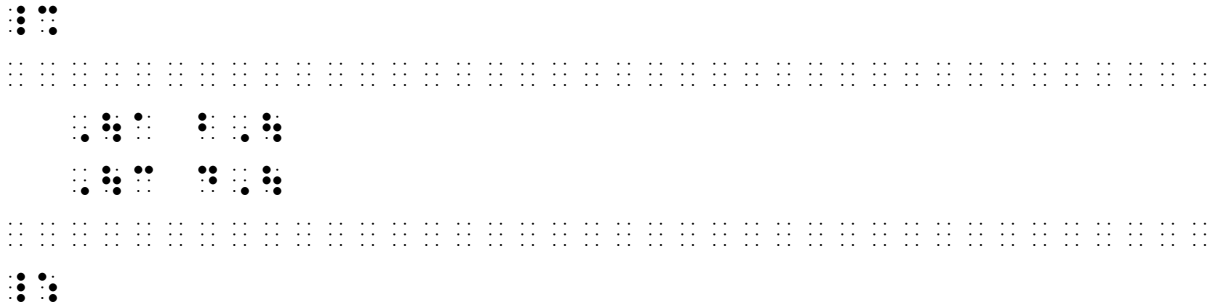
Print observation: The numerical entries are centered in their columns.



Braille observations: The numeric indicator is used for each number. Items begin in the same cell in each column (entries are left-aligned within each column). One blank cell separates the widest entry in a column from the beginning of the next column. The widest entry in the last column determines the placement of the right (closing) enlarged grouping signs. Enlarged brackets enclose this two-row array.

The English letter indicator is not used with any letter or combination of letters in an array.

Example 16.13-2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$



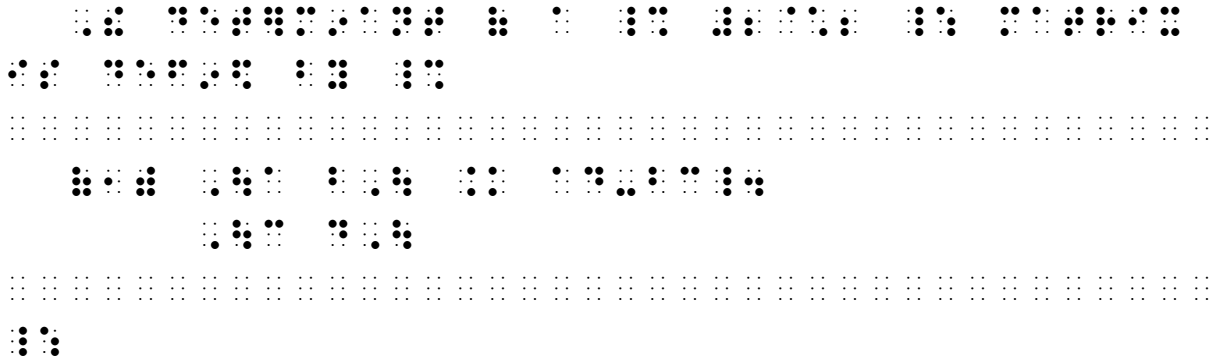
Each English letter is brailled without a letter indicator. Enlarged vertical bars enclose this two-row array.

16.13.5 Placement of Identifiers, Symbols, and Punctuation: Material outside of an array (such as identifiers, punctuation, signs of operation, or signs of comparison) are placed on the top line of the arrangement even though the material may be centered in print.

Example 16.13-3 The determinant of a 2×2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \qquad (1)$$

Print observations: An equals sign and expression are centered to the array. A period ends the statement. The expression is identified to the right with a numeral in parentheses.

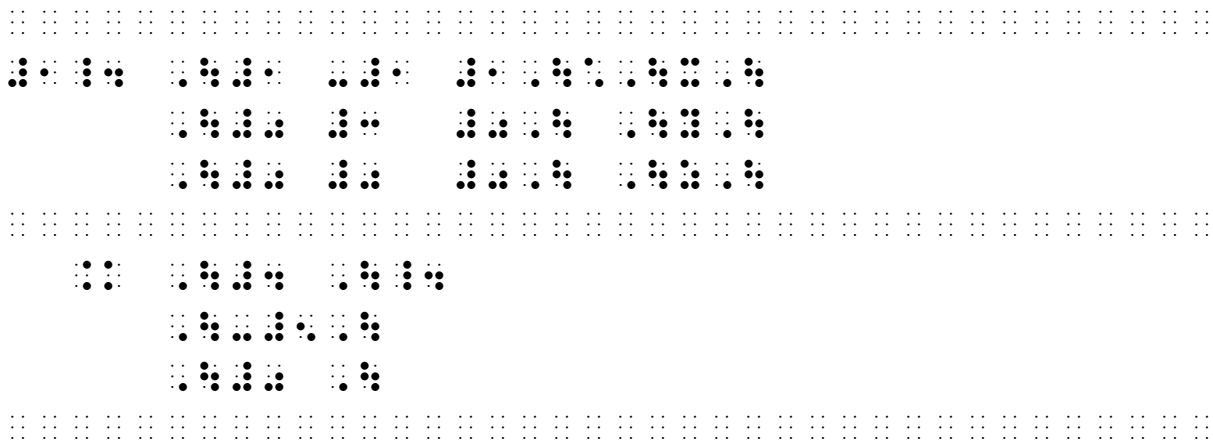


*Braille observations: The material outside of the determinant is placed on the top line of the arrangement. The identifier is moved to the left of the displayed expression according to the rules of the Nemeth Code. (See **Lesson 11,** "Displayed Material with Labels".)*

Example 16.13-4

$$1. \quad \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 4 \\ -5 \\ 0 \end{vmatrix} .$$

Print observations: An item number (1.) is centered to the 3×3 *array. Also centered within the arrangement are a multiplication dot, an equals sign, and a period.*



Braille observations: The material outside of the array is placed on the top line of the arrangement. The multiplication dot is unspaced from the items being multiplied, following spacing rules for operation signs. The entire expression will not fit across the braille page so it is divided before the comparison sign as dictated by the rules of the Nemeth Code.

PRACTICE 16F

1. $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

2. $\begin{pmatrix} 1 & -\frac{4}{3} & \frac{5}{3} \\ 2 & 5 & 12 \end{pmatrix}$

3. $\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \end{bmatrix}$

4. $\begin{vmatrix} ab & cd \\ ac & ce \end{vmatrix}$

5. Explain why points $(a_1,\ b_1), (a_2,\ b_2),$ and $(a_3,\ b_3)$ are collinear if and only if

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

6. The unit vectors in the direction of the x, y, and z axes of a three dimensional Cartesian coordinate system are

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \text{and}\ \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Further Considerations with Determinants and Matrices

In order to save space, code switch indicators are omitted from further examples unless both UEB and Nemeth Code are in use. Assume there is a blank line before and after each arrangement.

16.14 Multiplying Arrays: Recall that a multiplication problem can be printed without a multiplication symbol. When each factor is enclosed in grouping signs, it is understood that the side-by-side factors are to be multiplied. For example, $3 \cdot 2$ and $(3)(2)$ both mean "three times two". Similarly, when arrays are being multiplied, the multiplication symbol is often not printed. This array

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

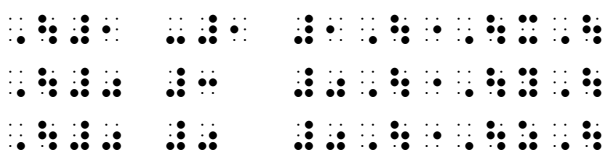
can also be printed without the multiplication dot, like this:

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

The spacing between the arrays shown (in print) is there only to distinguish between the two factors. The space is not brailled. When vertical bars are used, a multipurpose indicator is brailled between the two symbols, otherwise it is misread as a double vertical bar.

Example 16.14-1

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$



The multipurpose indicator is brailled between each side-by-side enlarged vertical bar, on every line of the arrangement.

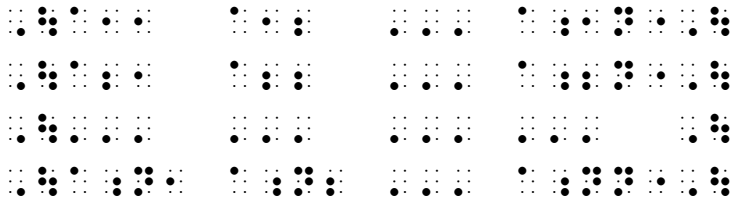
16.15 Omission Dots: When dots are printed in an array to indicate omission of one or more rows or columns, a series of dot 3s is used in braille according to the following observations.

16.15.1 No Dots Are Printed Between Columns: When at least one dot is printed in each column and no dots are printed between columns, each omission is brailled as a 3-cell Nemeth Code ellipsis beginning in the leftmost cell of its column.

Example 16.15-1

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Print observations: Omissions reading down some columns are printed as one dot; omissions reading across some rows are printed as three dots.



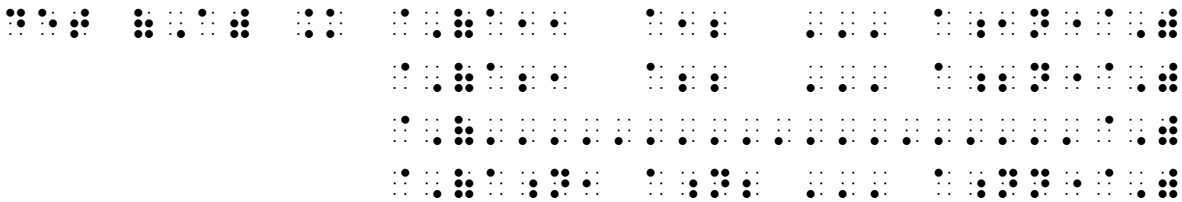
Braille observation: A baseline indicator is needed to assure the closing grouping signs on lines 1, 2, and 4 are on the same level as the opening grouping signs.

16.15.2 Dots Are Printed Between Columns or Some Entries are Blank: Here is the same problem, printed in another style where dots are strung completely across the omitted row (row 3). Notice that the dots occupy space between the columns. A row of omission dots is brailled across the full width of the array.

Example 16.15-2

$$\det(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Print observation: The third row is printed as a series of dots across the width of the array.



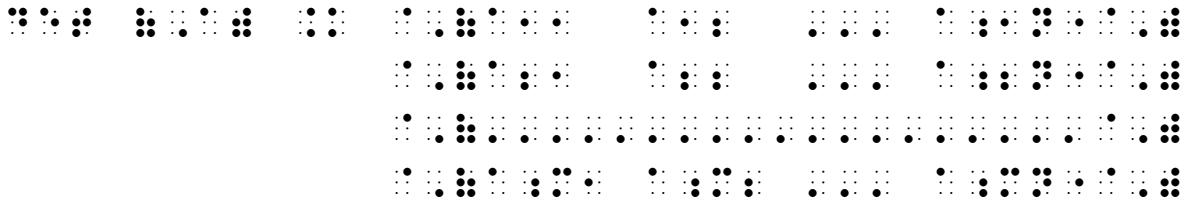
Braille observations: The row of dots is brailled as a series of unspaced dot 3s, beginning in the first cell of the first column and extending to the end of the longest entry in the last column. On lines 1, 2, and 4 a baseline indicator is needed to assure each closing grouping sign is on the same level as the opening grouping sign.

Another print style shows the missing items in the outer columns as single dots, and the missing items in columns 2 and 3 as blank space.

Example 16.15-3

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

In braille, the same method shown in the previous example is used for this print style. Only one omission row is brailled across the full width of the array.



PRACTICE 16G

In the next equation, matrix Y is expressed as the product of matrices B and X.

$$\begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ y_r \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ b_{31} & b_{32} & \cdots & b_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{vmatrix} \quad \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ x_n \end{vmatrix}$$

CHALLENGE PROBLEM #2

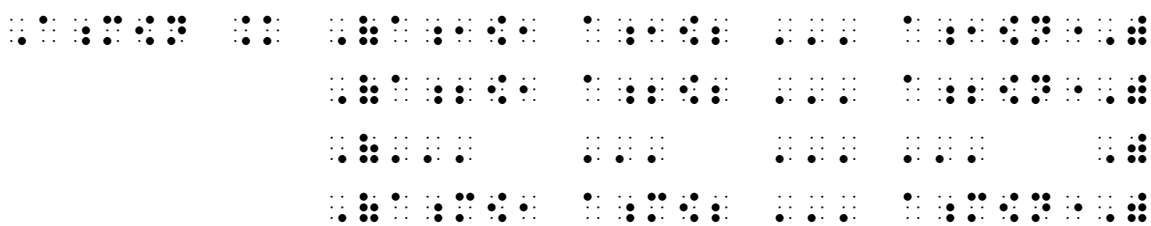
Omissions in an array may also be printed as depicted below.

$$A_{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Print observations: Omissions reading down some columns are printed as three stacked dots (a "vertical" ellipsis); omissions reading across some rows are printed as three dots; one blank entry is printed as an oblique ellipsis.

Which braille rule applies?

As long as at least one dot is printed in each column and no dots are printed between columns, each omission is brailled as a 3-cell Nemeth Code ellipsis beginning in the leftmost cell of its column.



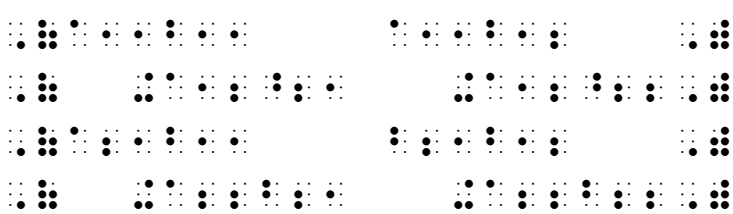
Review rules regarding the contracted comma in ***Lesson 8****.*

16.16 Space-Saving Techniques: Sometimes space-saving techniques are required in order to confine the arrangement to one braille page.

16.16.1 Runovers With Indentation: Entries may be run over to a new line. Each runover is indented two cells from the first cell of the first line of the entry. Attempt to apply the priority list regarding division of long expressions, but those rules can be disregarded if there is no other way around the problem. There is no need to explain the runover format to the reader.

Example 16.16-1

$$\begin{pmatrix} \mathbf{a}_{11}\mathbf{b}_{11}+\mathbf{a}_{12}\mathbf{b}_{21} & \mathbf{a}_{11}\mathbf{b}_{12}+\mathbf{a}_{12}\mathbf{b}_{22} \\ \mathbf{a}_{21}\mathbf{b}_{11}+\mathbf{a}_{22}\mathbf{b}_{21} & \mathbf{b}_{21}\mathbf{b}_{12}+\mathbf{a}_{22}\mathbf{b}_{22} \end{pmatrix}$$

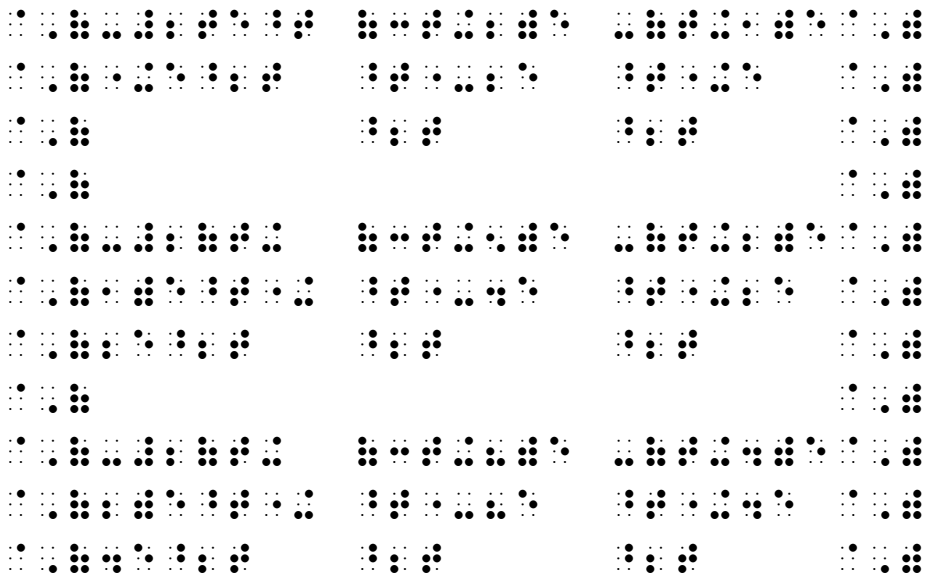


This "runovers with indentation" method is the preferred space-saving technique.

16.16.2 Runovers Without Indentation: If the technique above is not feasible, entries may be run over to new lines without indentation. Preference rules for runovers of mathematical expressions need not be observed if space would be saved. In order to distinguish each row, a blank line is inserted between them. Enlarged grouping symbols are brailled on those blank lines within the arrangement.

Example 16.16-2

$$\left[\begin{array}{ccc} -2te^{t}+e^{2t} & (3t+2)e^{t}-2e^{2t} & -(t+1)e^{t}+e^{2t} \\ -2(t+1)e^{t}+2e^{2t} & (3t+5)e^{t}-4e^{2t} & -(t+2)e^{t}+2e^{2t} \\ -2(t+2)e^{t}+4e^{2t} & (3t+8)e^{t}-8e^{2t} & -(t+4)e^{t}+4e^{2t} \end{array}\right]$$

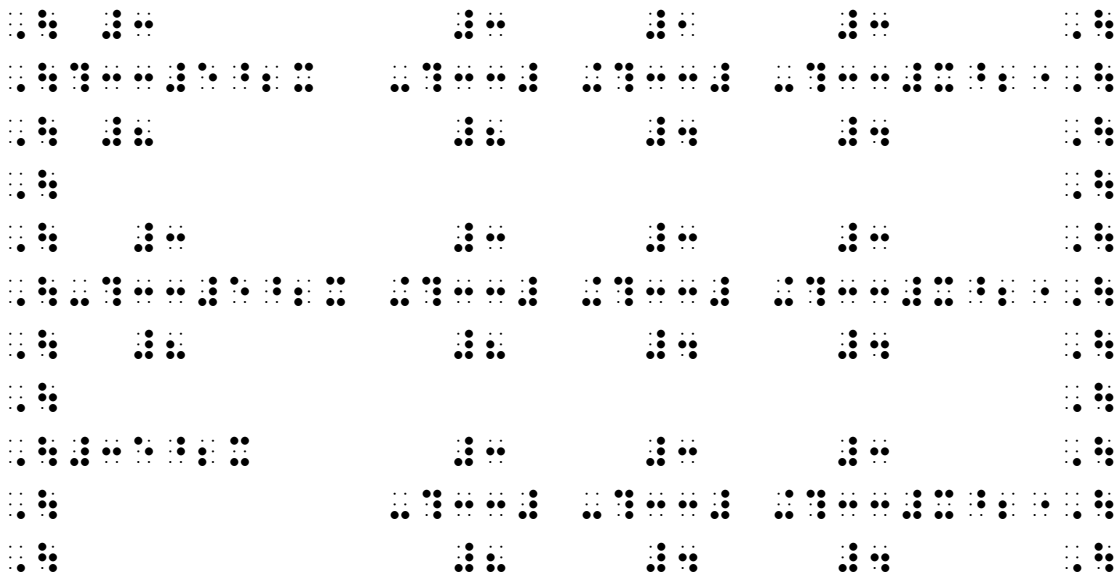


16.16.3 Fractions in Arrays: Fractions may be shown spatially if linear fractions take up too much space. Observe the following details.

- A line is skipped between the rows containing the spatial fraction. Enlarged grouping symbols are brailled on those blank lines within the arrangement.
- Operation signs and variables are placed on the same line as the spatial fraction line.
- Each entry is moved as far up as possible in its row. This includes entries that are not fractions.
- If the last item in a row is a superscript, or a subscript that uses a subscript indicator, a return to the baseline must occur before the closing grouping symbol is brailled. If the row extends fully to the closing grouping symbol, a baseline indicator is required to return to the baseline. If the row does not extend to the closing grouping symbol, the space returns the reader to the baseline.

Example 16.16-3

$$\begin{vmatrix} \frac{3}{8}e^{2x} & -\frac{3}{8} & +\frac{1}{4} & -\frac{3}{4}x^2 \\ -\frac{3}{8}e^{2x} & +\frac{3}{8} & +\frac{3}{4} & +\frac{3}{4}x^2 \\ 3\ e^{2x} & -\frac{3}{8} & -\frac{3}{4} & +\frac{3}{4}x^2 \end{vmatrix}$$



The single-line entry in row 3, column 1, is brailled on the first line of the row even though it is centered in its row in print.

16.16.4 Keying: When no other method saves the required space, the technique of keying discussed in the next lesson should be used.

Instructions: In the first array, use spatial fractions. In the second array, use runovers with indentation. Reminder: The partial derivative symbol ∂ was introduced in **Lesson 13**.

(*Note:* The image appears dark but it is not printed in boldface.)

PRACTICE 16H

The derivative of T can be expressed as follows.

$$dT = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

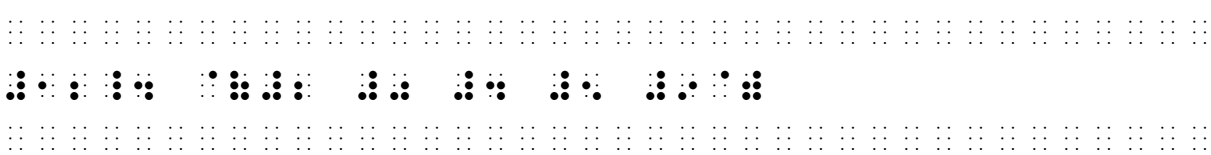
The Jacobian determinant of the mapping is shown below.

$$J(\rho, \theta, \phi) = \begin{vmatrix} \cos\theta \sin\phi & \sin\theta \sin\phi & \cos\phi \\ -\rho \sin\theta \sin\phi & \rho \cos\theta \sin\phi & 0 \\ \rho \cos\theta \cos\phi & \rho \sin\theta \cos\phi & -\rho \sin\phi \end{vmatrix}$$

$$= -\rho^2 \sin\phi$$

16.17 Row Matrix: A row matrix has only one row. Because it is a matrix, it is brailled as a spatial arrangement—that is, a line is left above and below the arrangement. Regular grouping symbols are used, not enlarged symbols, despite the larger appearance of the grouping symbols in print. Here is an example of a 1×5 row matrix.

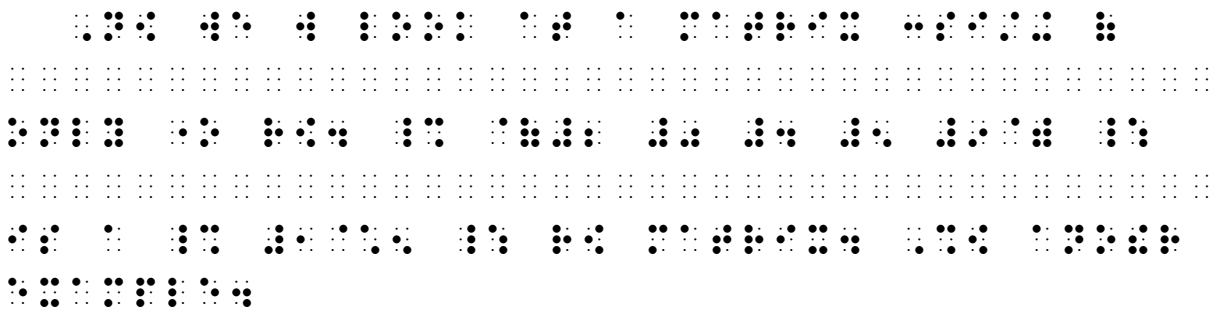
Example 16.17-1 12. $\begin{bmatrix} 2 & 0 & 4 & 5 & 9 \end{bmatrix}$



Observation: This is not an "enclosed list" because there are no commas between items. Context will make it clear that this is a row matrix.

16.18 Embedded Arrays: If the array is embedded within narrative text, the required blank line is inserted on the line before and the line after the line where the arrangement lies. Text is placed on the same line as the array if it fits.

Example 16.18-1 Now we will look at a matrix consisting of only one row. $\begin{bmatrix} 2 & 0 & 4 & 5 & 9 \end{bmatrix}$ is a 1×5 row matrix. Show another example.

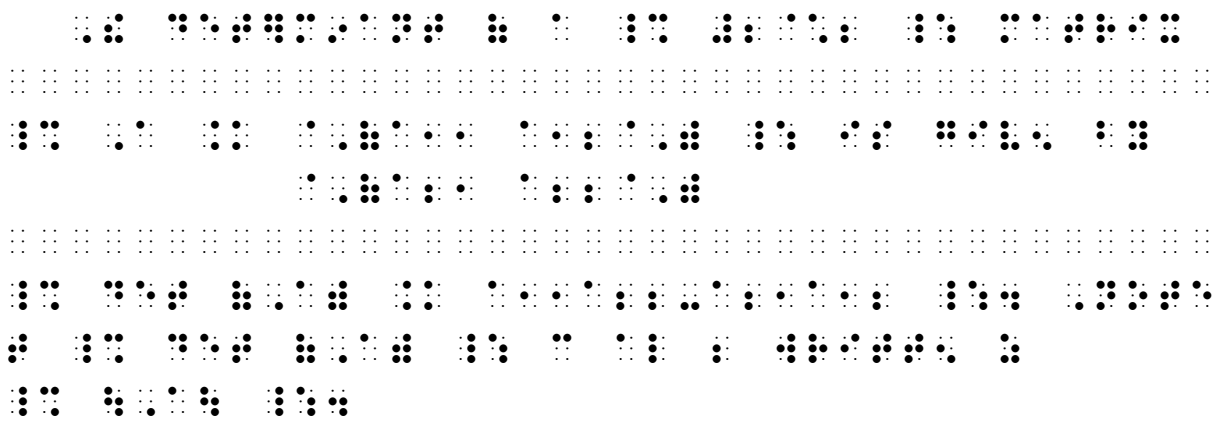


If the embedded array consists of more than one line, the surrounding text is placed on the top line of the arrangement only – before and/or after the top row of the array, as it fits in context of the narrative. Narrative continues in the runover cell of the paragraph. The opening Nemeth Code indicator and the Nemeth Code terminator are placed only on the top line of the embedded arrangement. The code switch indicators apply to the whole arrangement.

Example 16.18-2 The determinant of a 2×2 matrix $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$\det(A) = a_{11}a_{22} - a_{21}a_{12}$. Note that $\det(A)$ can also be written as $|A|$.

Print observation: The two rows of the embedded matrix are vertically centered to the equals sign in print.



*Braille observations: The first row of the embedded matrix is brailled on the same line as "*A =*". The second row is aligned beneath the first row; no text is brailled on that line.*

Note that "det" is a function abbreviation and therefore is followed by a space in the transcription.

PRACTICE 16I

1) If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & -6 \end{bmatrix}$, find AB.

   Answer: $AB = \begin{bmatrix} 1 & -25 & 10 \\ 29 & 1 & -18 \end{bmatrix}$.

2) Here are three examples of matrix operations.

   a) $\begin{bmatrix} 5 & 4 \end{bmatrix} + \begin{bmatrix} 20 & 30 \end{bmatrix} = \begin{bmatrix} 25 & 34 \end{bmatrix}$

   b) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 7 \\ 1 & 9 \end{bmatrix}$

   c) $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \end{bmatrix}$

16.19 Use of Tactile Graphics for Enlarged or Horizontal Grouping Signs: Enlarged grouping symbols may be drawn in place of the braille equivalents, especially when space saving is a factor or when a horizontal grouping sign is required. The horizontal brace shown below is best presented as a tactile graphic. Refer to BANA's *Guidelines and Standards for Tactile Graphics* for drawing techniques.

$$(2x+3)(x-5) = 2x^2 \underbrace{-10x + 3x}_{} - 15$$

$$= 2x^2 \quad -7x \quad -15$$

For further practice, see Appendix A—Reading Practice.

ANSWERS TO PRACTICE MATERIAL

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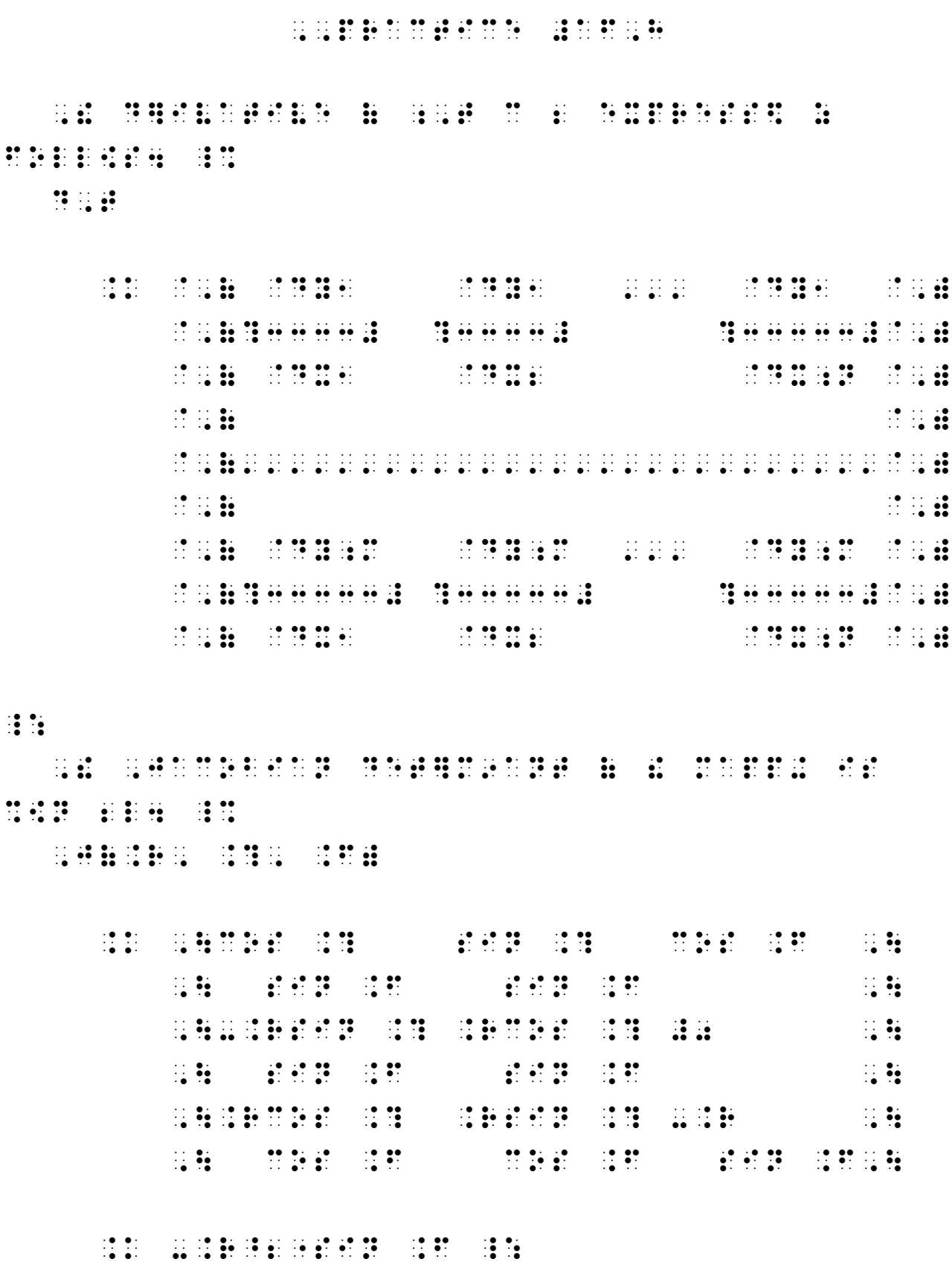

Note that blank lines are placed before and after the spatial portions of the problems only.

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EXERCISE 16

Exercise 16 will be available when this course is finished being written and is no longer "Provisional".

Proceed to Lesson 17.