

## **LESSON 12**

- MODIFIERS AND MODIFIED EXPRESSIONS
  - Common Modifiers
  - Binomial Coefficient
  - Modified Expressions and Superscripts / Subscripts
  - Modified Signs of Comparison
  - Expressions with More Than One Modifier

### *Format*

- Formal Proof

### *Answers to Practice Material*

## **LESSON PREVIEW**

Another type of mathematical notation with vertical components is studied. This lesson also applies the "mathematical statement" format to formal proofs.

## MODIFIERS AND MODIFIED EXPRESSIONS

### [NC Rule 15]

Some mathematical notation has a vertical aspect that challenges linear braille notation. By using modified expression indicators, the transcriber can relay the material in a compact manner, and the reader can construct the symbols into a meaningful expression.

#### 12.1 Definition

A modifier is a symbol or a combination of symbols occurring *directly over* or *directly under* its related symbol or expression. Here are some typical examples.

$\overset{?}{=}$  a question mark over an equals sign

$\underline{3.15}$  an underlined digit

$1.\overline{37}$  a line over two numerals (signifying a repeating decimal)

$\overleftrightarrow{AB}$  an arrow over two letters (signifying a line)

$\hat{k}$  a caret over a letter

#### 12.2 Construction of Simple Modified Expressions – The Five-Step Rule

An expression modified using the five-step method is initiated and terminated with special indicators.

$\ddot{\cdot}$	Multipurpose Indicator
$\ddot{\circ}$	Termination Indicator

The position of the modifier (above or below the expression) is also shown with the use of an indicator.

$\overset{\cdot}{\cdot}$	Directly-Over Indicator
$\underset{\cdot}{\cdot}$	Directly-Under Indicator

The process of constructing a modified expression is known as "The Five-Step Rule".

- (1) The *multipurpose indicator* is placed immediately before the expression to be modified.
- (2) The *expression* to be modified is written second.

(3) The *directly-over indicator* or the *directly-under indicator* is written third to show the position of the modifier.

(4) The *modifying symbol* is written fourth.

(5) The *termination indicator* is written last to show the end of the modification.

Notice that the multipurpose indicator, dot 5, signals the beginning of the modified expression and the termination indicator signals the completion of the modified expression.

Prototype for an expression with a modifier printed directly above it:  $\cdot \underline{\quad} \ddot{\cdot} \underline{\quad} \ddot{\cdot}$

Prototype for an expression with a modifier printed directly below it:  $\cdot \underline{\quad} \ddot{\cdot} \underline{\quad} \ddot{\cdot}$

All components of an expression modified according to the five-step rule should not be divided between lines. If that is not possible, strategies will be presented in Lesson 15.

### *Common Modifiers*


The most commonly used modifiers are presented in this lesson. You have seen many of these symbols in other contexts in previous lessons. Symbols of the code not shown here may also be used as modifiers.


#### 12.3 Arrows as Modifiers


Any of the arrows of the Nemeth Code may be a modifier. Those used in this section are shown below.

$\ddot{\cdot} \ddot{\cdot}$	Arrow barbed at right, contracted form	$\rightarrow$
$\ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot}$	Arrow barbed at both ends	$\leftrightarrow$
$\ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot}$	Arrow shaft with hollow dot at right	$\rightarrow\circ$
$\ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot}$	Upper barb only, right pointing	$\rightarrow$
$\ddot{\cdot} \ddot{\cdot} \ddot{\cdot} \ddot{\cdot}$	Arrow barbed at right, dashed shaft	$\dashrightarrow$


12.3.1 **Right-Pointing Arrow.** A right-pointing arrow in regular type with a full barb and single shaft of ordinary length is transcribed in its contracted form when used as a modifier above or below a math expression. The shaft length in print is determined by the width of the expression it modifies. If the modified expression is wide, the arrow shaft will be long in print, but this is not considered to be a "longer than ordinary" arrow shaft in this context.

$\Rightarrow \overrightarrow{YZ}$ 

  
*YZ is the expression being modified. It is modified directly over with a right-pointing arrow.*


$\Rightarrow \underline{T}$ 

  
*T is the expression being modified. It is modified directly under with a right-pointing arrow.*


$\Rightarrow \overrightarrow{M'N'}$ 

  
*The contracted form of the right-pointing arrow is used. The arrow shaft covers a wide expression and so is longer in print.*


Show exactly what portion of an expression is being modified. In the next example, only the expressions between the grouping symbols are modified. The grouping symbols are placed outside of the multipurpose and termination indicators.

$\Rightarrow |\overrightarrow{O_1P}| + |\overrightarrow{P_1P_2}|$ 
  

  
 *$O_1P$  and  $P_1P_2$  are the expressions being modified. Each is modified directly over with a right-pointing arrow.*

12.3.2 **Other Arrows.** All other types of arrows require the appropriate arrow symbol.

$\Rightarrow \overleftrightarrow{AB}$ 

  
*AB is the expression being modified. It is modified directly over with an arrow barbed at both ends.*

$\Rightarrow \overrightarrow{\circ}AB$ 

  
*AB is the expression being modified. It is modified directly over with an arrow with a hollow dot at its right end, no barb.*

$\Rightarrow \overrightarrow{\text{OT}}$ 

  
*OT (boldface) is the expression being modified. It is modified directly over with an arrow with an upper right-pointing barb.*

$\rightrightarrows$   $\overrightarrow{OB}$

*OB is the expression being modified. It is modified directly over with a right-pointing arrow with a dashed shaft.*

Example 12-1

$\leftrightarrow$   
 PQ is a line through points P and Q.

$\leftrightarrow$

Example 12-2

We now conclude that  $\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP}$

$\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP}$

*Reminder from Lesson 1: If a math expression will fit on one line but there is not room for one or both of the switch indicators, one or both switch indicators may stand alone on a line. Keeping the mathematical expression intact on one line is the priority.*

12.3.3 **When to Omit Arrows.** When identical arrows are used above vectors in boldface type throughout the text, they are omitted from the braille transcription. The boldface font attribute of the vectors is considered to be sufficient identification.

Example 12-3

**Vector Addition** By adding vectors  $\overrightarrow{CD}$  and  $\overrightarrow{EF}$ , the resultant vector,  $\mathbf{r}$ , is found.

$$\overrightarrow{CD} + \overrightarrow{EF} = \mathbf{r}$$

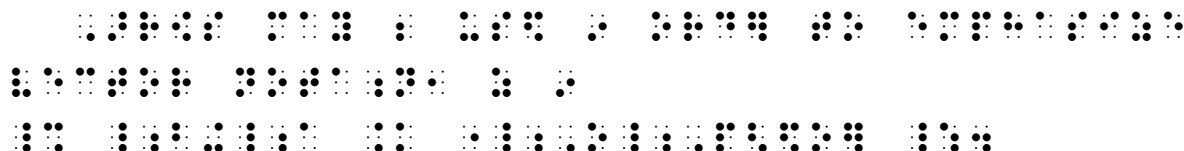
*The arrows printed above CD and EF are not transcribed. The bold typeform is retained.*

When the transcriber omits the vector arrows, a transcriber's note is required to explain the presence of the arrows in print. Sample transcriber's note: "The right-pointing arrows printed over boldface vectors are omitted."

A notable exception occurs when the author specifically refers to the arrows as a notational device. In that case, the arrow is transcribed in addition to the bold typeform.

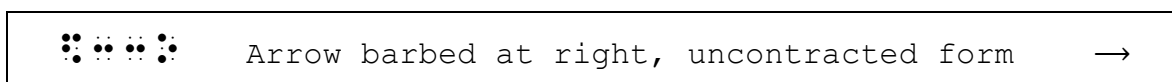
Example 12-4

Arrows may be used in order to emphasize vector notation, as in  $\mathbf{b} + \mathbf{a} = \overrightarrow{\mathbf{OP}}$ .



*The arrow is transcribed because it is mentioned in the narrative.*

12.3.4 **When the Arrow Itself is Being Modified.** A right-pointing arrow in regular type with a full barb and single shaft of ordinary length is transcribed in its uncontracted form when the arrow itself is being modified.



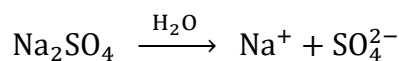
*A right-pointing arrow is the expression being modified.*

*It is modified directly over with the operation  $f \circ g$*

*Reminder: In function notation,  $\circ$  is a hollow dot, not the letter "o".*

12.3.5 **When Other Rules Apply.** Some arrangements that may appear to be modified arrows are not transcribed using the Five-Step Rule.

- Signs of Comparison Compounded Vertically.** Horizontal arrows printed one below the other are a sign of comparison compounded vertically. Review this topic in Lesson 9.
- Chemistry.** Words or symbols printed above reaction arrows are not treated as modifiers. Rules for these constructions are found in *Chemical Notation Using the Nemeth Braille Code*.



- Elementary Mathematics—"ARROW MATH" in Lower Grades.** These arrows are not "modified arrows". They are not being used as a sign of comparison, but rather as a "process". Draw these arrows as a tactile graphic. Refer to BANA's *Guidelines and Standards for Tactile Graphics* for techniques.

Example 12-5

Use arrow math to add  $36 + 23$ .

$36 \xrightarrow{+20} 56 \xrightarrow{+3} 59 \quad 36 + 23 = 59$

*Instructions:* Review arrow construction in Lesson 9. *Reminder:* In braille, identifiers for nonspatial itemized material must all begin on a new line in cell 1.

In the sentence at the end of this practice, assume that all vectors in the document are shown using that particular arrow notation. Show the proper way to omit the vector arrows in the transcription. Include the required transcriber's note after the topic heading.

**PRACTICE 12A**

Here are two modified arrows:  $x \overset{g}{\rightarrow} y \overset{f}{\rightarrow} z$

**Arrows as Modifiers**

1.  $\overleftarrow{F}$
2.  $\overleftarrow{AB}$
3.  $\overleftarrow{CD}$
4.  $\overleftrightarrow{OB} \cup \overleftrightarrow{OC}$
5.  $\overset{\circ}{\overrightarrow{EF}}$
6.  $\overleftarrow{T}$
7.  $\overleftrightarrow{XZ} \parallel \overleftrightarrow{RS}$
8.  $\overleftrightarrow{AB} + \overleftrightarrow{CD}$

**Vector Addition**

$\mathbb{R}$  equals  $\overrightarrow{OP}$  equals  $\overrightarrow{OM}$  plus  $\overrightarrow{MC}$  plus  $\overrightarrow{CP}$ .







➤  $\bar{\lambda}_t$     ⠠⠠⠠⠠⠠⠠

Example 12-9

The  $\bar{7}$ 's are repeated.

⠠⠠⠠    ⠠⠠⠠    ⠠⠠⠠⠠⠠⠠⠠⠠⠠    ⠠⠠⠠    ⠠⠠⠠    ⠠⠠⠠⠠⠠⠠⠠⠠⠠

*The apostrophe-s relates to the numeral, therefore it is transcribed inside the switches. The punctuation indicator prevents the apostrophe from being misread as a prime sign.*

Example 12-10

<u>Arabic</u>	<u>Roman</u>
5,000	$\bar{V}$
10,000	$\bar{X}$
50,000	$\bar{L}$
100,000	$\bar{C}$

⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠⠠⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠⠠⠠⠠
⠠⠠	
⠠⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
⠠⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
⠠⠠	

*The Arabic numbers in the first column can be transcribed in either UEB or Nemeth. The Roman numerals in column two require a code switch because they are modified. Smoother reading is accomplished by transcribing both columns in Nemeth. Review simple table format in Lesson 5.*

- b. **Bar Below a Single Digit or Letter.** When a single digit or a single letter (in any typeform or any alphabet) is modified only by a single horizontal bar directly under it, the directly-under indicator followed by the symbol for the bar is placed immediately after the digit or letter modified.

⠠⠠ Directly-Under Indicator followed by  
Horizontal Bar

- $x \underline{R}^+$  ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠
- $\underline{3} \underline{579}$  ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠

Study the examples below. Notice when the contracted form of the bar can be used. All simulated braille is in Nemeth. Switch indicators are omitted.

Example 12-11

- (1)  $(\bar{x}, \bar{y})$  ⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠
- (2)  $\bar{x}$ 's and  $\bar{y}$ 's ⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠
- (3)  $\overline{x + y}$  ⠠⠠⠠⠠⠠⠠⠠⠠
- (4)  $\underline{x + y}$  ⠠⠠⠠⠠⠠⠠⠠⠠
- (5)  $A(\underline{sn})$  ⠠⠠⠠⠠⠠⠠⠠⠠
- (6)  $\frac{\overline{PR}}{\overline{OR}}$  ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠
- (7)  $P(\bar{x}) = \overline{P(x)}$  ⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

---

*Instructions:* Treat the three examples of unit vectors in problem #2 as displayed mathematical material.

## PRACTICE 12B

### Carets and Bars as Modifiers

1. Unit vectors can be denoted with normal vector notation,  $\mathbf{u}$  or  $\vec{u}$ , or with standard unit vector notation,  $\hat{\mathbf{u}}$ , spoken "u-hat".
2. Unit vectors in various coordinate systems use Greek and English letters.

Cartesian coordinate system:  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$

Cylindrical coordinate system:  $\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}$

Spherical coordinate system:  $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$

3.  $1.142857\overline{142857}$

4.  $\frac{7}{15} = .4\overline{6}$

5.  $2 \cdot 3 = \overline{2} \cdot 3 = \overline{2 \cdot 3}$

6.  $\overline{PQ}, \overline{x'}, \overline{R''S''}$

7.  $\overline{s}, \overline{\alpha}, \overline{m'}$

8.  $m\overline{BC} = a$

9.  $\overline{C} = 100 \times 1000$

10.  $F = 2\pi\overline{r}l$

11.  $P(\overline{a + bi}) = \overline{0} = 0$

12.  $\hat{x}_i = 0.5(\overline{x}_i + \underline{x}_i)$

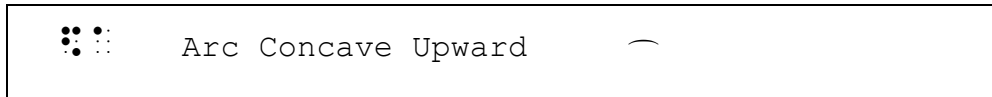
13.  $3.141\underline{59}$

---

## 12.6 Other Symbols Used as Modifiers

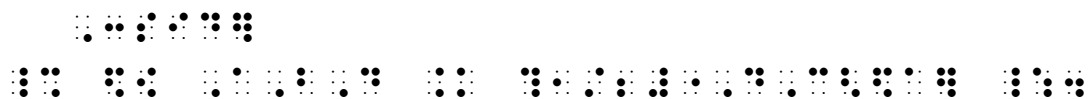
Apply the five-step rule for the construction of simple modified expressions when the following symbols are used as modifiers.

### 12.6.1 Arc

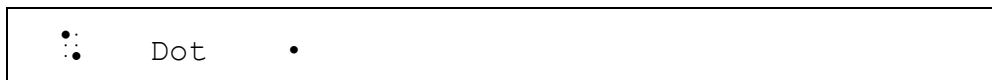


#### Example 12-12

Consider  $\angle ABD = \frac{1}{2} \widehat{DC}$ .



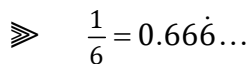
### 12.6.2 Dot



In print, the recurrence of one digit in a decimal numeral may be indicated by one dot over the recurring digit.



*"4" is the expression being modified. It is modified directly over with a single dot.*



*The last "6" is the expression being modified. It is modified directly over with a single dot.*

The recurrence of one or more digits in a decimal numeral may be indicated by one dot over each recurring digit. In braille, only one dot is used as a modifier. The dot is placed after the last modified digit in the recurring sequence.



Example 12-15

To simplify the polynomial, combine like terms.

$$(2x + 3)(x - 5) = 2x^2 - 10x + 3x - 15$$

$$= 2x^2 - 7x - 15$$

- b. **As a Modifier.** When a horizontal grouping sign is part of the math expression, it is regarded as a modifier. Although it is preferable to draw the symbol, the transcriber can represent the construction as shown below, following the five-step rule of modified expressions. The left grouping symbol is transcribed when the modifier is directly over the expression, and the right grouping symbol is transcribed when the modifier is directly under the expression.

$$\gg \overbrace{x + y} \quad \text{⠠⠠⠠⠠⠠⠠⠠⠠} \quad \underbrace{x + y} \quad \text{⠠⠠⠠⠠⠠⠠⠠⠠}$$

*A brace is printed above x + y.                      A brace is printed below x + y.*

- 12.6.4 **Question Mark.** When the question mark is not functioning as a sign of omission, the punctuation mark is transcribed. In mathematical context, a punctuation indicator precedes the question mark to prevent it from being misread as the numeral 8.

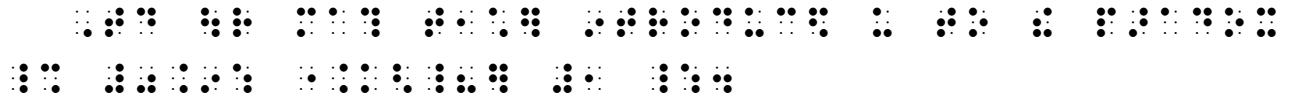
$\text{⠠⠠} \quad \text{Question Mark} \quad ? \quad (\text{not an omission})$
---

$$\gg \overset{?}{=} \quad \text{⠠⠠⠠⠠⠠⠠⠠⠠}$$

*An equals sign is the expression being modified. It is modified directly over with a question mark.*

Example 12-16

Today our math teacher introduced us to the paradox  $0.\overline{9} \stackrel{?}{=} 1$ .



12.6.5 Tilde

	Single Tilde	~
	Extended Tilde	~~~~

*The extended tilde has more than one peak.*

➤  $\tilde{u} = 0.8$

➤  $\tilde{A} \cup \tilde{B}$

➤  $\widetilde{r + s + t}$

The next example is a single tilde even though it is covering several items. (The single tilde has only one peak.)

➤  $\widetilde{r + s}$

12.7 Expressions as Modifiers

When a modifier is, itself, a mathematical expression, follow Nemeth Code rules regarding spacing and use of indicators. Each example below contains a comparison sign in the modifier. Nemeth rules applied include the following. A space is required before and after a sign of comparison; an English-letter indicator is not needed when a letter immediately precedes or follows a sign of comparison or when a letter touches an indicator; a numeric indicator is required when a numeral follows a space. The termination indicator marks the end of the modifier.

➤  $\lim_{x \searrow a}$

*lim is modified directly under with the expression  $x \searrow a$ .*

➤  $\bigcup_{A \subset F}$

*The union ("cup") symbol is modified under with the expression  $A \subset F$ .*

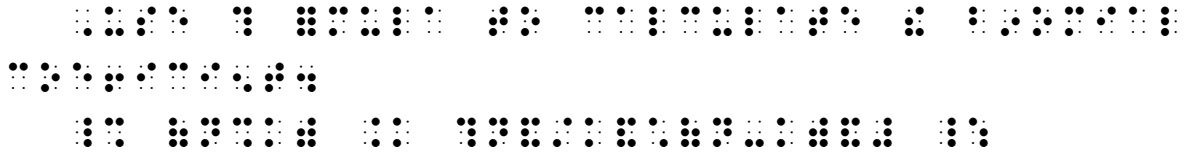




Example 12-18

Use this formula to calculate the binomial coefficient.

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$



---

**PRACTICE 12C**

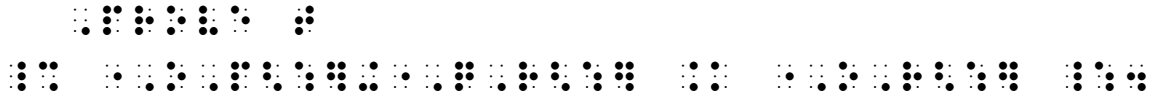
1. In Figure 7.3, if  $\widehat{AB} = \widehat{CD}$  in circle O, then  $\angle AOB = \angle BOC$ .
2.  $x \stackrel{?}{=} y$  means "does  $x$  equal  $y$ ?"
3.  $\sum_{d|n}$  (where  $d|n$  means " $d$  divides  $n$ ").
4.  $\binom{t}{p} = R_t^p$
5.  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  for all integers  $n \geq 0$ .
6. Does  $(\tilde{x}, \tilde{y})$  mean  $\tilde{x}$  and  $\tilde{y}$ ?
7. More modified expressions:
  - a.  $.249\grave{9}$
  - b.  $2.431\grave{3}\grave{1}$
  - c.  $\acute{x}\grave{y} - \grave{y}\acute{x}$
  - d.  $\grave{a} + \grave{\grave{a}} = ?$
  - e.  $f \rightarrow \tilde{f}$
  - f.  $\widetilde{x + y}$

## 12.8 Spacing with Modified Expressions

The spacing before and after an entire modified expression is subject to the spacing rules for the symbols preceding or following it.

### Example 12-19

Prove that  $\overline{OP} + \overline{QR} = \overline{OR}$ .



*There is no space before or after the operation (plus) sign. There is a space before and after the comparison (equals) sign.*

When the modifier is wider than the modified symbol, the print copy will insert extra space to clarify what exactly is being modified. In the next example, only the Sigma is modified, not the letter  $a$ . The space between the Sigma and the  $a$  clarifies the extent of the modifier,  $i < j$ . In braille, however, indicators define the extent of a modifier. There is no need for the space in braille.

$$\gg \sum_{i < j} a_{ij}$$

*Sigma is modified directly under with the inequality  $i < j$ .  
The termination indicator signals the completion of the modifier.*

## PRACTICE 12D

### Spacing with Modified Expressions

- A. The probability of the event A, written  $P(A)$ , is defined as

$$P(A) = \sum_{\wedge} f(x)$$

where  $\sum_{\wedge} f(x)$  means sum  $f(x)$  over those values  $x_i$  that are in A.

B.  $\times_{\alpha \in A} A_{\alpha}$

C.  $\prod_{i > j} (x_i - x_j)$

D.  $\overline{7} + 2 \stackrel{?}{=} \overline{7 + 2}$



$$\gg \overline{OD^2} + \overline{OP^2}$$



*The first dot 5 begins the first modified expression. The second dot 5 is a baseline indicator following the first superscript "2". Similarly, the third dot 5 begins the second modified expression and the fourth dot 5 is a baseline indicator following the second superscript.*

$$\gg \overline{x_1} \quad \dots$$

*You may wish to review Section 6.11 regarding nonuse of the subscript indicator.*

- a. **Binomial Coefficient.** Although a binomial coefficient is not technically a modified expression, notice how this rule applies.

$$\gg \binom{a_x}{b_y} \quad \dots$$

*The baseline indicator precedes the "directly under" indicator. This keeps that indicator on the same level of writing as the letters "a" and "b".*

## PRACTICE 12E

### Superscripts and Subscripts

- A)  $\overline{AB}^2 + \overline{BC}^2$
- B)  $\overline{A} = [\overline{a_i}]$
- C)  $\sqrt{\overline{\dot{x}^2} + \overline{\dot{y}^2}}$
- D)  $\underline{\underline{Z}}^\circ$
- E) If  $\overline{a_1} = 72$ , find  $\overline{a_7}$ .
- F) Draw  $\overline{P_1P_2}$  if  $P_1$  is the point (1, 3) and  $P_2$  is the point (2, -1).
- G)  $(\overline{3^{-1}}) \in P$
- H)  $\overline{x_1} + \overline{y_1}$

12.9.3 **Modified Expression on the Baseline That Follows a Superscript or a Subscript.** When a modified expression written on the baseline of writing immediately follows a superscript or a subscript, several details must be considered in order to determine the necessity of level indicators. Because braille dot 5 has several meanings besides that of baseline indicator (hence the name "multipurpose indicator") mindful use of that symbol is required in order to give the reader the correct information. Several examples will illustrate. Each dot-5 is highlighted in the examples.

- a. If a level indicator is needed for the superscript or subscript, the baseline indicator is transcribed before starting the modified expression.

⤵  $x^2\bar{y}$      ⠠⠨⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

*This dot 5 is a baseline indicator, following the superscript 2.*

⤵  $\bar{x}_n\bar{z}_m$      ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

*This dot 5 is a baseline indicator, following the subscript n.*

- b. If a subscript indicator is not needed, a baseline indicator is not needed before starting the modified expression.

⤵  $x_1\bar{z}_1$      ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

*Each 1 is printed at the subscript level. There are no dot 5s in this example because a baseline indicator is not needed when a subscript indicator is not used, and the contracted form of the "bar over" does not use a dot 5.*

⤵  $x_1\bar{\bar{z}}_1$      ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

*Each 1 is printed at the subscript level. A baseline indicator is not needed when a subscript indicator is not used. This dot 5 is the start of the modified expression.*

- c. Two indicators may be needed. First, a baseline indicator (dot 5) is used to terminate the effect of the superscript or subscript level indicator. Next, a multipurpose indicator (dot 5) is required to begin the five-step modification.

⤵  $3x^2\bar{\Delta x} + 3x\bar{\Delta x}^2$

⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

*Because  $x^2$  requires a superscript indicator and the five-step rule is used for the modified expression, two dot 5s are needed. The baseline*







## *Modified Signs of Comparison*

### 12.11 Definition

A modified sign of comparison consists of a simple sign of comparison, such as the equals sign or the tilde, modified by a caret, dot, triangle, question mark, vertical bar, or any symbol except another sign of comparison.

*When a simple sign of comparison occurs above or below another simple sign of comparison the combination is transcribed as a sign of comparison compounded vertically. See Section 5.8 for a review of that construction. Note that many of those signs are printed with a single horizontal line "bar over" or "bar under". For example, "bar over greater than, inclusion with bar under, bar over single tilde, logical sum with bar under," etc. These signs are not to be misinterpreted as a horizontal bar modifying a sign of comparison.*

### 12.12 Transcription

A modified sign of comparison as defined above is transcribed in accordance with the five-step rule for modified expressions.

In addition to the caret and inverted caret seen earlier in this lesson, you may also encounter a left- or right-pointing caret in a modified sign of comparison. Do not confuse these two symbols with the "less than" and "greater than" comparison signs. Ask an expert if context does not clarify the identity of this symbol.

⠠⠠⠠⠠⠠⠠	Left-Pointing Caret	<
⠠⠠⠠⠠⠠⠠	Right-Pointing Caret	>

The following list contains the modified equals signs most commonly used.

Modified Equals Sign		
⠠⠠⠠⠠⠠⠠⠠⠠	Caret Over Equals Sign	≐
⠠⠠⠠⠠⠠⠠⠠⠠	Caret Under Equals Sign	≑
	("is projective to")	^
⠠⠠⠠⠠⠠⠠⠠⠠	Inverted Caret Over Equals Sign	≒
⠠⠠⠠⠠⠠⠠⠠⠠	Left-Pointing Caret Over Equals Sign	≓
⠠⠠⠠⠠⠠⠠⠠⠠	Right-Pointing Caret Over Equals Sign	≔

⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Dot Over Equals Sign ("is approximately equal to")	$\doteq$
⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Dot Over and Dot Under Equals Sign	$\dot{=}$
⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Two Dots Over and Two Dots Under Equals Sign	$\ddot{=}$
⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Hollow Dot Over Equals Sign ("is equal in degrees to")	$\overset{\circ}{=}$
⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Equilateral Triangle Over Equals Sign	$\overset{\triangle}{=}$
⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Question Mark Over Equals Sign	$\overset{?}{=}$
⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨	Vertical Bar Over Equals Sign	$\overset{ }{=}$

➤  $\sqrt{5} \doteq 2.236$

⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨

➤  $5(3) + 2(-2) \overset{?}{=} 11$

⠠⠨

If a symbol is encountered which does not appear on this list, construct a symbol in accordance with the same principles. A transcriber-created symbol should be listed on the Special Symbols page.

➤  $A \overset{m}{=} B$     ⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨⠠⠨

The remainder of the list provided in the code contains other modified comparison signs most commonly used.

Modified Tilde		
	Dot Under Tilde	$\underset{\cdot}{\sim}$
	Dot Over Tilde	$\overset{\cdot}{\sim}$
Modified Horizontal Bar		
	Caret Over Bar	$\overset{\wedge}{-}$
	Caret Under Bar	$\underset{\wedge}{-}$
	("is perspective to")	
	Dot under Bar	$\bar{\cdot}$

If the horizontal bar is modified by a dot *over* it, the combination is a modified sign of operation ("minus with dot over" signifying "proper difference"). The five-step rule is not used for this symbol. See Section 5.4.7.

*Instructions:* Use the five-step rule to show the horizontal grouping sign in the last item.

## PRACTICE 12G

### Modified Signs of Comparison and More

- $A \doteq 3.14r^2$
- $\angle b \doteq \frac{1}{2} \widehat{EB}$
- The symbol  $\hat{=}$  is used to make a definition.
- $x \sim \mathcal{N}(0, 1)$
- $x^n = \underbrace{x \cdot x \cdot x \cdot \dots \cdot x}$

## *Expressions with More Than One Modifier*

### 12.13 Modifiers of Higher Order

When two or more modifiers occur one above the other and apply to exactly the same expression, the second, third, etc. modifiers are "modifiers of higher order." In such cases, the directly-over indicator or the directly-under indicator is doubled, tripled, etc., before each modifier to indicate its position. The termination indicator is used only after the last modifier shown.

Directly-over indicators	
⠠⠠	Second order
⠠⠠⠠	Third order
Directly-under indicators	
⠨⠨	Second order
⠨⠨⠨	Third order
⠨	Termination indicator

$$\Rightarrow \frac{a=3}{x+y}$$



*Analysis: The expression  $x + y$  is modified directly over with a horizontal bar which, in turn, is modified directly over with the expression  $a = 3$ .*

$$\Rightarrow \frac{x+y}{\frac{a=3}{b=2}}$$



*Analysis: The expression  $x + y$  is modified directly under with a horizontal bar which, in turn, is modified directly under with the expression  $a = 3$  which is modified directly under with the expression  $b = 2$ .*

$$\gg \sum_{\substack{i,j,k \\ i < j < k}}$$



*Analysis: The Sigma is modified directly under with letters i, j, k . That, in turn, is modified directly under with the expression i < j < k. Review the rules regarding use of the English-letter indicator in Lesson 3. An English-letter indicator is needed for the first letter "j" because it is a "single letter" according to the definition in the Nemeth Code. The other letters are either touching indicators or are preceded or followed by a sign of comparison, so an English-letter indicator is not needed.*

**12.13.1 Parallel Horizontal Bars.** When two or more parallel horizontal bars are the same length and apply to exactly the same expression, they are treated as a single modifier. In such cases, the directly-over or the directly-under indicator is used only once, and the symbol for the bar is used as many times as necessary to correspond to the print text.

$$\gg \overline{9} \quad \dots$$

$$\gg (\overline{A}) \quad \dots$$

$$\gg \overline{\overline{x}} \quad \dots$$

*Attention:* Two parallel horizontal bars must not be misinterpreted as the equals sign, and three parallel horizontal bars must not be misinterpreted as the identity sign. The symbol's meaning can be determined by reading the surrounding text for context. See Section 5.8 ("Signs of Comparison Compounded Vertically") to review equals sign printed above or below a sign of comparison.

## 12.14 Individual Modifiers

When two or more modifiers do not apply to exactly the same symbols but cover different portions of the same expression, the longer modification encloses the entire modified expression. Within the long expression, each inner expression is modified individually.

$$\gg \overline{\overline{\overline{x} \times \overline{\overline{x}}}} \quad \dots$$

*Analysis: In the simulated braille below, the longer modification indicators and symbols are highlighted; the inner modified expressions are underlined.*





As you study the eight constructions in the next example, note the use or nonuse of the English-letter indicator and the use or nonuse of the numeric indicator. This reminder applies to the first item. *Reminder from Lesson 8: When a linked expression will not fit on one braille line, start a new line with the link (the comparison sign).*

Example 12-20

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad (1)$$

⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠  
 ⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

$$\sum_{k=1}^n (2k) = n(n + 1) \quad (2)$$

⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

$$\sum_{i,j=1}^n a_{ij} x_i x_j \quad (3)$$

⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

$$\sum_{x_1=0}^1 \sum_{x_2=0}^1 \quad (4)$$

⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

$$\bigcup_{r=1}^n A_r = A_1 \cup A_2 \dots \quad (5)$$

⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

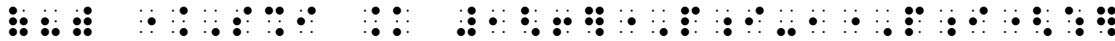
$$\overline{\prod_{x \in a} c^x} \quad (6)$$

⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

$$\frac{\frac{b=2}{x+y}}{\frac{a=3}{a=3}} \quad (7)$$

⠠⠠⠠⠠ ⠠⠠

$$\sum_{i=1}^6 \overline{P_{i-1}P_i} \quad (8)$$



## PRACTICE 12H

### Expressions with More Than One Modifier

1.  $\overline{\overline{9} \cdot \overline{3}} = \overline{\overline{9} \cdot \overline{3}}$

5.  $\overline{x_{\overline{a}} + y^{\overline{n}}}$

2.  $\overline{\overline{A^n}}$

6.  $\overline{\overline{N}}$

3.  $\overline{\overline{A \cap B \cap C}}$

7.  $\sum_{i=1}^k \sum_{j=1}^k$

4.  $\overline{a_{\overline{n}} + b_{\overline{p}}}$

8.  $\prod_{\substack{j=1 \\ j \neq k}}^n$



## *Format*

### **12.16 Formal Proof [NC Rule 26.7]**

A proof is a valid argument that establishes the truth of a mathematical statement. It is often introduced by a heading such as *Theorem*, *Proposition*, or *Lemma*. A progressive sequence of statements leads to the conclusion. In a formal proof, every step of the argument is shown and each step is supported by a definition or by a previously proven statement.

Lesson 11 explained how to transcribe a mathematical statement. Those guidelines are summarized in steps a-c, below, in the context of a proof.

- a. A blank line precedes the beginning of the proof.
- b. *Heading*: The heading can be formatted as a paragraph heading or as a cell-5 or cell-7 heading, at the transcriber's discretion. Review Section 11.38.b for details.
- c. *Statement*: Continue with the text, using normal (3-1) paragraph style. When the statement is printed in a variant typeform and the proof follows, in regular type, it is recommended that the typeform be preserved for the statement in order to retain distinction.
- d. *Auxiliary Captions*: Paragraph headings such as Given, Hypothesis, Prove, or Conclusion begin in cell 3, without a blank line line before the paragraph. Associated material follows the caption. Runovers go in cell 1.
- e. *Two-Column Proof*: See [12.16.1](#), below.
- f. *End of Proof Icon*: See [12.16.2](#), below.
- g. When the proof is complete, insert a blank line before continuing with the text.

**12.16.1 Two-Column Proof.** When a formal proof is presented by numbered steps printed in two columns, the layout is changed as follows.

- a. If there is a caption such as "Proof", follow the same pattern established in [step d](#), above ("Auxiliary Captions").
- b. The column format is changed to a list in braille. A transcriber's note must call attention to the change in format. See [step c](#), below, for a sample transcriber's note.

A blank line is inserted before the list. If there are column headings, such as "Statement" and "Reason", see [step c](#), below. Each step begins in cell 1, starting with the first item from the left column. Runovers are in cell 3. The related item from the right column begins in cell 1 on the next line, with runovers in cell 3.

Each item must be labeled with an identifier as described below.

- c. *Step Identifiers*: Typically, the print copy includes the column headings "Statements" and "Reasons". In braille, the column headings are replaced with a letter – "S" or "R", respectively – as part of each step number. For example, "1S" is Statement 1, followed on the next line by "1R" which is Reason 1. If other column headings appear in print, the transcriber chooses an appropriate letter to represent each heading. If no column headings appear in print, use "S" for the left-column items and "R" for the right-column items.

If the steps are not numbered in print, identifiers are added in the manner stated above. A transcriber's note must specify the meaning of the letter labels. Any transcriber-added numbers or letters must also be explained.

Sample transcriber's note: "Proofs printed with steps in columns headed *Statements* and *Reasons* are transcribed with an S or R immediately following the step number to show the column in which the step appears. Each step from the *Statements* column is immediately followed by the corresponding step from the *Reasons* column."

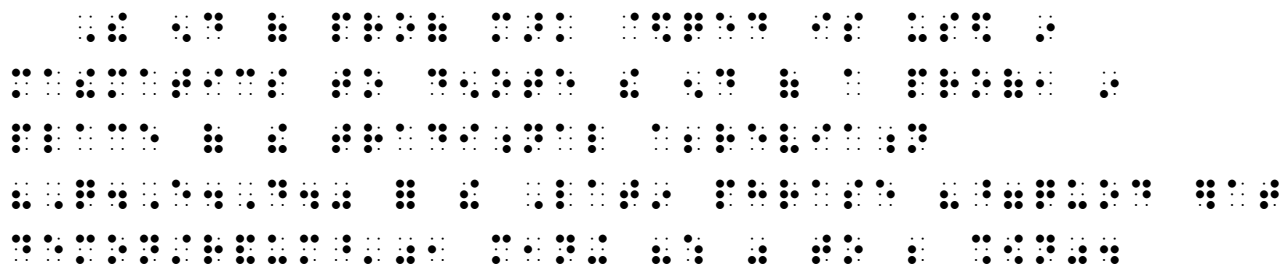
Identifiers are transcribed in either code (UEB or Nemeth) according to code-switching guidelines.

- 12.16.2 **End of Proof Icon**. Sometimes a shape (often a bold rectangle or square) or the Latin abbreviation "Q.E.D." is used to denote the end of a proof. The shape is often printed at the right margin and may go unnoticed. Watch for it. Regardless of the shape or abbreviation used in print, the icon shown below is used in braille, and is transcribed following the last word or character in the proof, preceded by one blank cell. The icon is created using the UEB transcriber-defined shape indicator and may be used in either Nemeth Code or UEB without switching. The icon must be listed on the Special Symbols page.



Example 12-21

The end of proof mark ■ is used in mathematics to denote the end of a proof, in place of the traditional abbreviation "Q.E.D." for the Latin phrase "**quod erat demonstrandum**", meaning "which was to be shown".



*Contractions are used in the Latin words according to Braille Formats.*

Example 12-22

(Assume the required transcriber's note regarding the two-column proof appears on the Transcriber's Notes page.)

**THEOREM 2. All right angles are equal.**

*Given:*  $\angle ABC$  and  $\angle DEF$  are right angles.

*Prove:*  $\angle ABC$  equals  $\angle DEF$ .

*Proof:*

<u>Statements</u>	<u>Reasons</u>
1. $\angle ABC$ and $\angle DEF$ are right angles.	1. Given.
2. $\angle ABC = 90^\circ$ , $\angle DEF = 90^\circ$ .	2. A right angle contains 90 degrees.
3. $\angle ABC = \angle DEF$ .	3. Transitivity postulate. ■

7  
8  
9  
10  
11  
12  
13  
14  
15  
16  
17  
18  
19  
20  
21  
22  
23  
24

Line 7: A blank line precedes the proof.

Line 8: The heading is printed in capital letters and also in a nonregular typeface (boldface).

When a paragraph heading is printed in capital letters, typeform is disregarded.

Lines 8-9: The paragraph style is 3-1. Distinctive typeform (boldface) is retained in the statement.

Lines 10-11, 12-13, 14: Each auxiliary caption follows print regarding typeform (italics, in this example), and uses a 3-1 paragraph style.

Line 15: A blank line precedes the list.

Lines 16-23: Each item in the 2-column proof begins in cell 1, with runovers in cell 3. Identifying letters S and R are combined with each step number.

Line 23: A dark square is printed in the right margin to mark the end of the proof. The "qed" icon is transcribed.

Line 24: A blank line follows the proof.

---

## PRACTICE 12I

*Instructions:* Create a Transcriber's Notes page that might appear in a volume which contains the proof shown in Practice 12J. A statement citing the code book is required in every volume that uses the Nemeth Code. In the first paragraph, state the title and edition of the Nemeth code book as well as any Updates. Something like this:

Mathematical content is transcribed according to *The Nemeth Braille Code for Mathematics and Science Notation, 2022*.

In the second paragraph, explain the step-number format as described in 12.16.1.c. Refer to *Braille Formats* for further guidelines regarding the structure of a Transcriber's Notes page.

## PRACTICE 12J

**Given:**  $3x = 7 - \frac{1}{2}x$

**To Prove:**  $x = 2$

STEP	REASON
1. $3x - 7 - \frac{1}{2}x$	1. GIVEN
2. $6x = 14 - x$	2. Multiplication Property
3. $7x = 14$	3. Addition Property
4. $x = 2$	4. Division Property

*For further practice, see Appendix A—Reading Practice.*

## **EXERCISE 12**

Prepare Exercise 12 for your grader.

















