

LESSON 12

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LESSON PREVIEW

Another type of mathematical notation with vertical components is studied. This lesson also applies the "mathematical statement" format to formal proofs.

Some of the print examples are enlarged in order to show the modifiers more clearly on the printed page. You may also wish to use a magnifier to remove any uncertainty.

MODIFIERS AND MODIFIED EXPRESSIONS

[NC Rule 15]

Some mathematical notation has a vertical aspect that challenges linear braille notation. By using modified expression indicators, the transcriber can relay the material in a compact manner, and the reader can construct the symbols into a meaningful expression.

12.1 Definition

A modifier is a symbol or a combination of symbols occurring *directly over* or *directly under* its related symbol or expression. Here are some typical samples.

$\overset{?}{=}$	a question mark over an equals sign
$\underline{3.15}$	an underlined digit
$1.\overline{37}$	a line over two numerals (signifying a repeating decimal)
\overleftrightarrow{AB}	an arrow over two letters (signifying a line)
\hat{k}	a caret over a letter

12.2 Construction of Simple Modified Expressions – The Five-Step Rule

An expression modified using the five-step method is initiated and terminated with special indicators.

$\ddot{\cdot}$	Multipurpose Indicator
$\ddot{\cdot}$	Termination Indicator

The position of the modifier (above or below the expression) is also shown with the use of an indicator.

$\overset{\cdot}{\cdot}$	Directly-Over Indicator
$\underset{\cdot}{\cdot}$	Directly-Under Indicator

The process of constructing a modified expression is known as "The Five-Step Rule".

- (1) The *multipurpose indicator* is placed immediately before the expression to be modified.
- (2) The *expression* to be modified is written second.
- (3) The *directly-over indicator* or the *directly-under indicator* is written third to show the position of the modifier.

(4) The *modifying symbol* is written fourth.

(5) The *termination indicator* is written last to show the end of the modification.

Notice that the multipurpose indicator, dot 5, signals the beginning of the modified expression and the termination indicator signals the completion of the modified expression.

Prototype for an expression with a modifier printed directly above it: ⠠_____⠨⠠_____⠨

Prototype for an expression with a modifier printed directly below it: ⠠_____⠨⠠_____⠨

All components of an expression modified according to the five-step rule should not be divided between lines. If that is not possible, strategies will be presented in Lesson 15.

Common Modifiers


The most commonly used modifiers are presented in this lesson. You have seen many of these symbols in other contexts in previous lessons. Symbols of the code not shown here may also be used as modifiers.

12.3 Arrows as Modifiers


Any of the arrows of the Nemeth Code may be a modifier. Those used in this section are shown in the box below.

⠠⠠	Arrow barbed at right, contracted form	→
⠠⠠⠠⠠	Arrow barbed at both ends	↔
⠠⠠⠠⠠⠠	Arrow shaft with hollow dot at right	→
⠠⠠⠠⠠⠠	Upper barb only, right pointing	→
⠠⠠⠠⠠	Arrow barbed at right, dashed shaft	→


12.3.1 **Right-Pointing Arrow.** A right-pointing arrow in regular type with a full barb and single shaft of ordinary length is transcribed in its contracted form when used as a modifier above or below a math expression. The shaft length in print is determined by the width of the expression it modifies. If the modified expression is wide, the arrow shaft will be long in print, but this is not considered to be a "longer than ordinary" arrow shaft in this context.

$\Rightarrow \overrightarrow{YZ}$


YZ is the expression being modified. It is modified directly over with a right-pointing arrow.


$\Rightarrow \underline{\rightarrow T}$


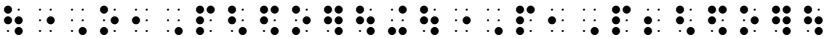
T is the expression being modified. It is modified directly under with a right-pointing arrow.

$\Rightarrow \overrightarrow{M'N'}$


The contracted form of the right-pointing arrow is used. The arrow shaft covers a wide expression and so is longer in print.


Show exactly what portion of an expression is being modified. In the next sample, only the expressions between the grouping symbols are modified. The grouping symbols are placed outside of the multipurpose and termination indicators.

$\Rightarrow |\overrightarrow{O_1P}| + |\overrightarrow{P_1P_2}|$





 O_1P and P_1P_2 are the expressions being modified. Each is modified directly over with a right-pointing arrow.


12.3.2 **Other Arrows.** All other types of arrows require the appropriate arrow symbol.

$\Rightarrow \overleftrightarrow{AB}$



AB is the expression being modified. It is modified directly over with an arrow barbed at both ends.

$\Rightarrow \overrightarrow{\circ AB}$


AB is the expression being modified. It is modified directly over with an arrow with a hollow dot at its right end, no barb.

$\Rightarrow \overrightarrow{\text{OT}}$


OT (boldface) is the expression being modified. It is modified directly over with an arrow with an upper right-pointing barb.

$\Rightarrow \overrightarrow{\text{OB}}$


OB is the expression being modified. It is modified directly over with a right-pointing arrow with a dashed shaft.

Example 12-1

\overleftrightarrow{PQ} is a line through points P and Q.

\overleftrightarrow{PQ} is a line through points P and Q.

Example 12-2

We now conclude that $\vec{OP} = \vec{OT} + \vec{TP}$

We now conclude that $\vec{OP} = \vec{OT} + \vec{TP}$

Reminder from Lesson 1: If a math expression will fit on one line but there is not room for one or both of the switch indicators, one or both switch indicators may stand alone on a line. Keeping the mathematical expression intact on one line is the priority.

12.3.3 Arrows Used in Vector Notation. When identical arrows are used above vectors in boldface type throughout the text, they are omitted from the braille transcription. The boldface font attribute of the vectors is considered to be sufficient identification. A transcriber’s note is required to explain the presence of the arrows in print. Sample transcriber’s note:

The right-pointing arrows printed over boldface vectors are omitted.

Example 12-3

Vector Addition. By adding vectors \vec{CD} and \vec{EF} , the resultant vector, \mathbf{r} , is found.

$$\vec{CD} + \vec{EF} = \mathbf{r}$$

1 $\vec{CD} + \vec{EF} = \mathbf{r}$
2 $\vec{CD} + \vec{EF} = \mathbf{r}$
3 $\vec{CD} + \vec{EF} = \mathbf{r}$
4 $\vec{CD} + \vec{EF} = \mathbf{r}$

Lines 2 and 4: The arrows printed above *CD* and *EF* are not transcribed. The bold typeform is retained. (Assume the required transcriber’s note appears on the Transcriber’s Notes page.)

See the [Postscript](#) at the end of this lesson for notes regarding other types of vector notation.

12.3.4 **When the Arrow Itself is Being Modified.** A right-pointing arrow in regular type with a full barb and single shaft of ordinary length is transcribed in its uncontracted form when the arrow itself is being modified.

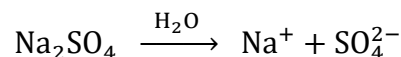
⋮ ⋮ ⋮ ⋮	Arrow barbed at right, uncontracted form	→
---------	--	---

⋮ ⋮ ⋮ ⋮ $x \xrightarrow{f \circ g} y$ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

*A right-pointing arrow is the expression being modified.
 It is modified directly over with the operation $f \circ g$
 Reminder: In function notation, \circ is a hollow dot, not the letter "o".*

12.3.5 **When Other Rules Apply.** Some arrangements that may appear to be modified arrows are not transcribed using the Five-Step Rule.

- a. **Signs of Comparison Compounded Vertically.** Horizontal arrows printed one below the other are a sign of comparison compounded vertically. Review this topic in Lesson 9.
- b. **Chemistry.** Words or symbols printed above reaction arrows are not treated as modifiers. Rules for these constructions are found in *Chemical Notation Using the Nemeth Braille Code*.



- c. **Elementary Mathematics—"ARROW MATH" in Lower Grades.** These arrows are not "modified arrows". They are not being used as a sign of comparison, but rather as a "process". Draw these arrows as a tactile graphic. Refer to BANA's *Guidelines and Standards for Tactile Graphics* for techniques.

Example 12-4

Use arrow math to add $36 + 23$.

$$36 \xrightarrow{+20} 56 \xrightarrow{+3} 59 \quad 36 + 23 = 59$$

1 ⠠⠼⠢⠸ ⠠⠼⠬⠸ ⠠⠼⠫⠸ ⠠⠼⠤⠸ ⠠⠼⠡⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸

2 ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸

3 ⠠⠼⠠⠸ ⠠⠼⠠⠸

4 ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸

5 ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸

6 ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸ ⠠⠼⠠⠸

Instructions: Review arrow construction in Lesson 9. *Reminder:* In braille, identifiers for nonspatial itemized material must all begin on a new line in cell 1.

In the sentence at the end of this practice, assume that all vectors in the document are shown using that particular arrow notation. Show the proper way to omit the vector arrows in the transcription. Include the required transcriber’s note after the topic heading.

PRACTICE 12A

Here are two modified arrows: $x \xrightarrow{g} y \xrightarrow{f} z$

Arrows as Modifiers

- 1. \overleftarrow{F}
- 2. \overleftarrow{AB}
- 3. \overleftarrow{CD}
- 4. $\overrightarrow{OB} \cup \overrightarrow{OC}$
- 5. $\overset{\circ}{EF}$
- 6. \overleftarrow{T}
- 7. $\overleftrightarrow{XZ} \parallel \overleftrightarrow{RS}$
- 8. $\overleftrightarrow{AB} + \overleftrightarrow{CD}$

Vector Addition

\mathbb{R} equals \overrightarrow{OP} equals \overrightarrow{OM} plus \overrightarrow{MC} plus \overrightarrow{CP} .

$$\gg \$11.\underline{25} \quad \dots\dots\dots$$

The 2-digit number 25 is modified directly under with a horizontal bar. Note that the decimal point is not part of this modified expression. The bar is under the digits only.

$$\gg 1.\overline{37} \quad \dots\dots\dots$$

The 2-digit number 37 is modified directly over with a horizontal bar.

Example 12-6

Prove that $\overline{OP} + \overline{QR} = \overline{OR}$.

$$\dots\dots\dots$$

Example 12-7

The fraction $\frac{3}{7}$ expressed as a decimal is .4285721428571428571 Written as a repeating decimal, $\frac{3}{7} = \overline{.428571}$.

$$\begin{array}{l} 1 \quad \dots\dots\dots \\ 2 \quad \dots\dots\dots \\ 3 \quad \dots\dots\dots \\ 4 \quad \dots\dots\dots \\ 5 \quad \dots\dots\dots \end{array}$$

The decimal point is not part of the modified expression. The multipurpose indicator is transcribed in the cell immediately preceding the numeral 4.

12.5.2 Bar Over or Bar Under Only One Digit or Letter. When a single digit or a single letter in any typeform or any alphabet is modified only by a single horizontal bar either directly over or directly under it, the five-step rule is not applied. Instead, a contracted form is used.

- a. **Bar Above a Single Digit or Letter.** When a single digit or a single letter (in any typeform or any alphabet) is modified only by a single horizontal bar directly over it, the symbol for the bar is placed immediately after the digit or letter modified.

$$\gg \overline{x} \quad \dots\dots$$

$$\gg \overline{3} \quad \dots\dots$$

$$\gg \overline{c}' \quad \dots\dots$$

⊳ $\bar{\lambda}_t$ ⠠⠠⠠⠠⠠⠠

Example 12-8

The $\bar{7}$'s are repeated.

⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

The apostrophe-s relates to the numeral, therefore it is transcribed inside the switches. The punctuation indicator prevents the apostrophe from being misread as a prime sign.

Example 12-9

<u>Arabic</u>	<u>Roman</u>
5,000	\bar{V}
10,000	\bar{X}
50,000	\bar{L}
100,000	\bar{C}

1	⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠⠠⠠⠠⠠
2	⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠⠠⠠⠠⠠
3	⠠⠠	
4	⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
5	⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
6	⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
7	⠠⠠⠠⠠⠠⠠⠠⠠	⠠⠠⠠⠠
8	⠠⠠	

The Arabic numbers in the first column can be transcribed in either UEB or Nemeth. The Roman numerals in column two require a code switch because they are modified. Smoother reading is accomplished by transcribing both columns in Nemeth. Review simple table format in Lesson 5.

- b. **Bar Below a Single Digit or Letter.** When a single digit or a single letter (in any typeform or any alphabet) is modified only by a single horizontal bar directly under it, the directly-under indicator followed by the symbol for the bar is placed immediately after the digit or letter modified.

⠠⠠ Directly-Under Indicator followed by
Horizontal Bar

➤ \underline{x} \underline{R}^+ ⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠

➤ $\underline{3}$ $\underline{579}$ ⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠

Study the seven expressions shown in [Example 12-10](#). Notice when the contracted form of the bar can be used. The simulated braille is in Nemeth. Code switch indicators are omitted.

Example 12-10

- (1) (\bar{x}, \bar{y}) ⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠
- (2) \bar{x} 's and \bar{y} 's ⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠
- (3) $\overline{x + y}$ ⠠⠠⠠⠠⠠⠠⠠⠠
- (4) $\underline{x + y}$ ⠠⠠⠠⠠⠠⠠⠠⠠
- (5) $A(\underline{sn})$ ⠠⠠⠠⠠⠠⠠⠠⠠
- (6) $\frac{\overline{PR}}{OR}$ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠
- (7) $P(\bar{x}) = \overline{P(x)}$ ⠠⠠⠠⠠⠠⠠⠠⠠ ⠠⠠⠠⠠ ⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

Instructions: Treat the three examples of unit vectors in problem #2 as displayed mathematical material.

PRACTICE 12B

Carets and Bars as Modifiers

1. Unit vectors can be denoted with normal vector notation, \mathbf{u} or \vec{u} , or with standard unit vector notation, $\hat{\mathbf{u}}$, spoken "u-hat".
2. Unit vectors in various coordinate systems use Greek and English letters.

Cartesian coordinate system: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$

Cylindrical coordinate system: $\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}$

Spherical coordinate system: $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$

3. $1.142857\overline{142857}$

4. $\frac{7}{15} = .4\overline{6}$

5. $2 \cdot 3 = \overline{2} \cdot 3 = \overline{2 \cdot 3}$

6. $\overline{PQ}, \overline{x'}, \overline{R''S''}$

7. $\overline{s}, \overline{\alpha}, \overline{m'}$

8. $m\overline{BC} = a$

9. $\overline{C} = 100 \times 1000$

10. $F = 2\pi\overline{r}l$

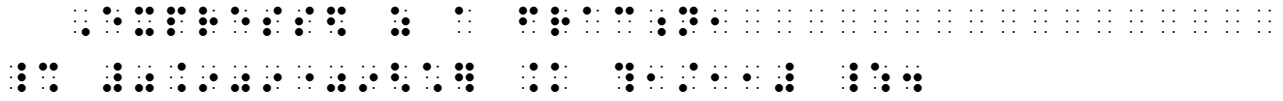
11. $P(\overline{a + bi}) = \overline{0} = 0$

12. $\hat{x}_i = 0.5(\overline{x}_i + \underline{x}_i)$

13. $3.141\overline{59}$

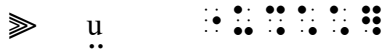
Example 12-12

Expressed as a fraction, $0.909\overset{\cdot}{0}\overset{\cdot}{9} = \frac{1}{11}$.



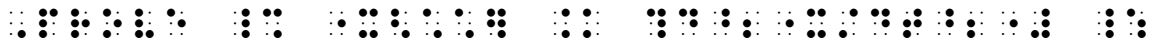
In print, a single dot is shown over each of the last two digits (0 and 9).

When one or more dots occur over or under a single letter or numeral, the symbol for the dot is used as many times as necessary to conform with the print text.



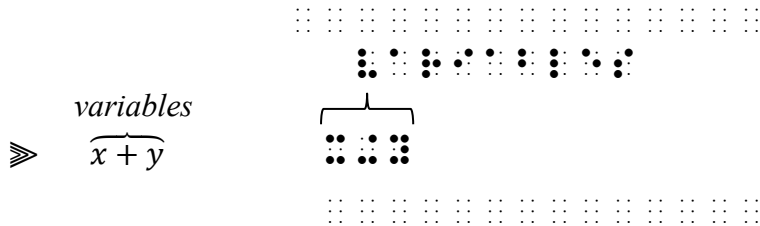
Example 12-13

Prove $\ddot{x} = \frac{d^2x}{dt^2}$



12.6.3 **Horizontal Grouping Sign.** When a horizontal grouping sign occurs over or under a mathematical expression, it is either a part of the expression or is pointing to a label.

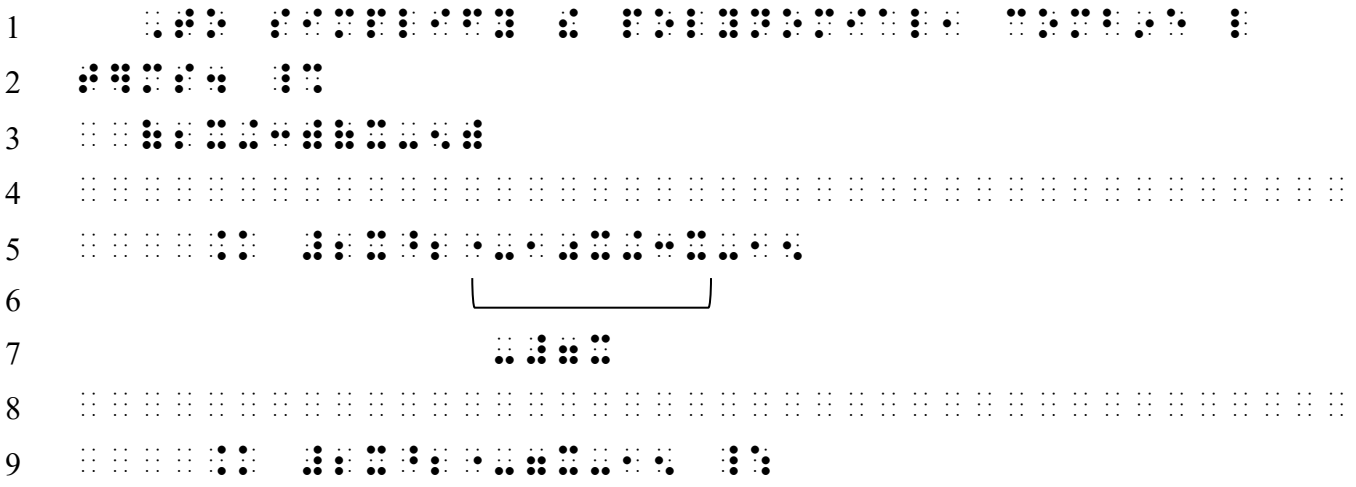
- a. **As a Pointer.** When the grouping sign points to a label or to explanatory text, it must be drawn as a tactile graphic. Refer to *Guidelines and Standards for Tactile Graphics* for drawing techniques.



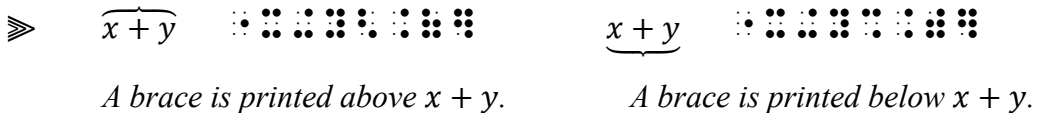
Example 12-14

To simplify the polynomial, combine like terms.

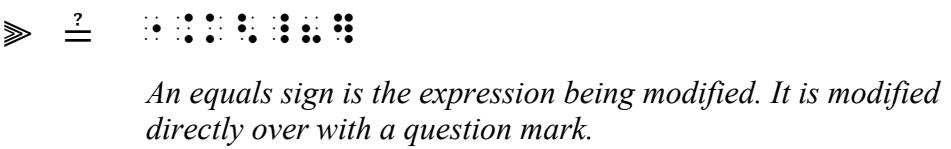
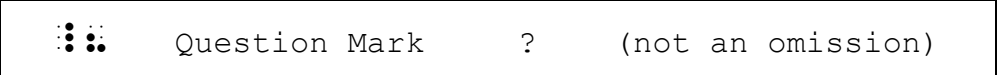
$$\begin{aligned} (2x + 3)(x - 5) &= 2x^2 - 10x + 3x - 15 \\ &= 2x^2 - 7x - 15 \end{aligned}$$



b. **As a Modifier.** When a horizontal grouping sign is part of the math expression, it is regarded as a modifier. Although it is preferable to draw the symbol, the transcriber can represent the construction as shown below, following the five-step rule of modified expressions. The left grouping symbol is transcribed when the modifier is directly over the expression, and the right grouping symbol is transcribed when the modifier is directly under the expression.

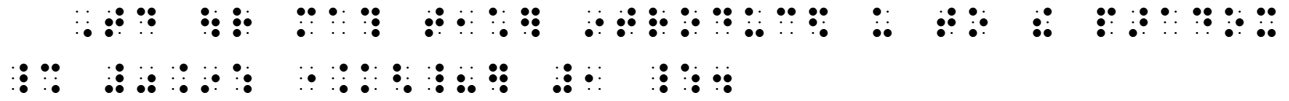


12.6.4 **Question Mark.** When the question mark is not functioning as a sign of omission, the punctuation mark is transcribed. In mathematical context, a punctuation indicator precedes the question mark to prevent it from being misread as the numeral 8.



Example 12-15

Today our math teacher introduced us to the paradox $0.\overline{9} \stackrel{?}{=} 1$.



12.6.5 Tilde

	Single Tilde	~
	Extended Tilde	⋈

The extended tilde has more than one peak.

- $\tilde{u} = 0.8$
- $\tilde{A} \cup \tilde{B}$
- $\overset{\sim}{r+s+t}$

The tilde in the next sample is a single tilde even though it covers several items. (The single tilde has only one peak.)

- $\widetilde{r+s}$

12.7 Expressions as Modifiers

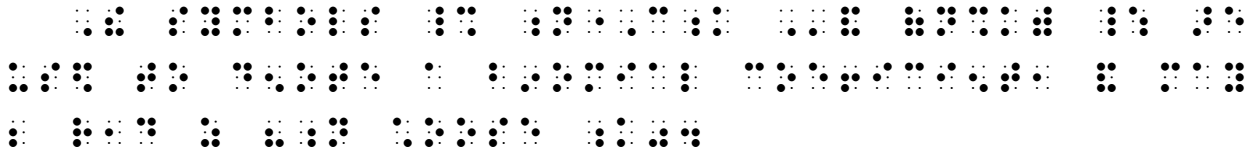
When a modifier is, itself, a mathematical expression, follow Nemeth rules regarding spacing and use of indicators. Here we show a comparison sign in each modifier. Nemeth rules applied include the following. A space is required before and after a sign of comparison; an English-letter indicator is not needed when a letter immediately precedes or follows a sign of comparison or when a letter touches an indicator; a numeric indicator is required when a numeral follows a space. The termination indicator marks the end of the modifier.

- $\lim_{x \searrow a}$
- lim is modified directly under with the expression $x \searrow a$.*

- $\bigcup_{A \subset F}$
- The union ("cup") symbol is modified under with the expression $A \subset F$.*

Example 12-17

The symbols ${}_nC_k$ and $\binom{n}{k}$ are used to denote a binomial coefficient, and may be read as "n choose k".



PRACTICE 12C

1. In Figure 7.3, if $\widehat{AB} = \widehat{CD}$ in circle O, then $\angle AOB = \angle BOC$.
2. $x \stackrel{?}{=} y$ means "does x equal y ?"
3. $\sum_{d|n}$ (where $d|n$ means " d divides n ").
4. $\binom{t}{p} = R_t^p$
5. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for all integers $n \geq 0$.
6. Does (\tilde{x}, \tilde{y}) mean \tilde{x} and \tilde{y} ?
7. More modified expressions:

a. $.249\dot{9}$	b. $2.431\dot{3}\dot{1}$	c. $\dot{x}\dot{y} - y\ddot{x}$
d. $\underset{\dots}{a} + \underset{\dots}{a} = ?$	e. $f \rightarrow \tilde{f}$	f. $\widetilde{x+y}$

12.8 Spacing with Modified Expressions

The spacing before and after an entire modified expression is subject to the spacing rules for the symbols preceding or following it.

Example 12-18

Prove that $\overline{OP} + \overline{QR} = \overline{OR}$.

$$\overline{OP} + \overline{QR} = \overline{OR}$$

There is no space before or after the operation (plus) sign. There is a space before and after the comparison (equals) sign.

When the modifier is wider than the modified symbol, the print copy will insert extra space to clarify what exactly is being modified. In the next sample, only the Sigma is modified, not the letter *a*. The space between the Sigma and the *a* clarifies the extent of the modifier, $i < j$. In braille, however, indicators define the extent of a modifier. There is no need for the space in braille.

$$\gg \sum_{i < j} a_{ij}$$

*Sigma is modified directly under with the inequality $i < j$.
The termination indicator signals the completion of the modifier.*

PRACTICE 12D

Spacing with Modified Expressions

- A. The probability of the event *A*, written $P(A)$, is defined as

$$P(A) = \sum_{\wedge} f(x)$$

where $\sum_{\wedge} f(x)$ means sum $f(x)$ over those values x_i that are in *A*.

- B. $\prod_{\alpha \in A} A_{\alpha}$
 C. $\prod_{i > j} (x_i - x_j)$
 D. $\overline{7} + 2 \stackrel{?}{=} \overline{7 + 2}$

➤ $\overline{OD^2} + \overline{OP^2}$



The first dot 5 begins the first modified expression. The second dot 5 is a baseline indicator following the first superscript "2". Similarly, the third dot 5 begins the second modified expression and the fourth dot 5 is a baseline indicator following the second superscript.

➤ $\overline{x_1}$ ⠠⠵⠗⠖⠎⠠

You may wish to review Section 6.11 in Lesson 6 regarding nonuse of the subscript indicator.

- a. **Binomial Coefficient.** Although a binomial coefficient is not technically a modified expression, notice how this rule applies.

➤ $\binom{a_x}{b_y}$ ⠠⠽⠞⠄⠠⠸⠎⠠⠽⠞⠄⠠

The baseline indicator precedes the "directly under" indicator. This keeps that indicator on the same level of writing as the letters "a" and "b".

12.9.3 Modifier Ends with a Superscript or Subscript. The terminator must be on the same level as the multipurpose indicator which starts the modified expression. When the last character(s) in the modifier is at the superscript or subscript level and a level indicator has been used, the baseline indicator precedes the terminator.

➤ median
 $(s, t) \in S_{xy}$



The first dot 5 begins the modification which is on the baseline of writing. The second dot 5 is a baseline indicator, which puts the terminator on the same level as the modified expression. (The membership symbol was introduced in Lesson 5.)

PRACTICE 12E

Superscripts and Subscripts

A) $\overline{AB}^2 + \overline{BC}^2$

B) $\overline{A} = [\overline{a}_i]$

C) $\sqrt{\dot{x}^2 + \dot{y}^2}$

D) \underline{Z}°

E) If $\overline{a}_1 = 72$, find \overline{a}_7 .

F) Draw $\overline{P_1P_2}$ if P_1 is the point $(1, 3)$ and P_2 is the point $(2, -1)$.

G) $(\overline{3^{-1}}) \in P$

H) $\overline{x}_1 + \overline{y}_1$

I) $\lim_{x \rightarrow 1^+}$

12.9.4

A Modified Expression on the Baseline That Follows a Superscript or a Subscript. When a modified expression written on the baseline of writing immediately follows a superscript or a subscript, several details must be considered in order to determine the necessity of level indicators. Because braille dot 5 has several meanings besides that of baseline indicator (hence the name "multipurpose indicator") mindful use of that symbol is required in order to give the reader the correct information. Several samples will illustrate. Each dot-5 is highlighted in the samples.

- a. If a level indicator is needed for the superscript or subscript, the baseline indicator is transcribed before starting the modified expression.

➤ $x^2\bar{y}$ ⠠⠵⠲⠠⠶

This dot 5 is a baseline indicator, following the superscript 2.

➤ $\bar{x}_n\bar{z}_m$ ⠠⠵⠽⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠⠠

This dot 5 is a baseline indicator, following the subscript n.

- b. If a subscript indicator is not needed, a baseline indicator is not needed before starting the modified expression.

➤ $x_1\bar{z}_1$ ⠠⠵⠠⠠⠠⠠⠠

Each 1 is printed at the subscript level. There are no dot 5s in this example because a baseline indicator is not needed when a subscript indicator is not used, and the contracted form of the "bar over" does not use a dot 5.

➤ $x_1\bar{\bar{z}}_1$ ⠠⠵⠠⠠⠠⠠⠠

Each 1 is printed at the subscript level. A baseline indicator is not needed when a subscript indicator is not used. This dot 5 is the start of the modified expression.

- c. Two indicators may be needed. First, a baseline indicator (dot 5) is used to terminate the effect of the superscript or subscript level indicator. Next, a multipurpose indicator (dot 5) is required to begin the five-step modification.

➤ $3x^2\bar{\Delta}x + 3x\overline{\Delta x^2}$

⠠⠢⠠⠵⠲⠠⠠

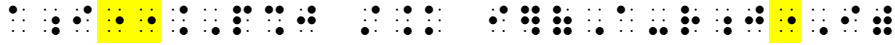
Because x^2 requires a superscript indicator and the five-step rule is used for the modified expression, two dot 5s are needed. The baseline indicator is immediately followed by the dot 5 which begins the modified expression.

➤ $a_k\overline{b_1 \dots b_p}$ ⠠⠠⠠

Because a_k requires a subscript indicator and the five-step rule is used for the modified expression, two dot 5s are needed. The baseline indicator is

immediately followed by a dot 5 beginning the modified expression. Finally, the directly-over indicator is preceded by a baseline indicator to ensure that the entire modified expression is transcribed on the same level.

$$\gg a_i \prod_{j \neq i} (A - r_j I)$$



Because a_i requires a subscript indicator and the five-step rule is used for the modified expression, two dot 5s are needed. The baseline indicator is immediately followed by a dot 5 beginning the modified expression. The last dot 5 is a baseline indicator following the subscript "j".

12.10 Modified Expression Within a Superscript or Subscript

Recall that a modifier and its related expression must be placed on the same level of writing. When a modified expression occurs as a superscript or subscript, or as the first part of a superscript or subscript, the level indicator is transcribed first, followed by the multipurpose indicator which begins the modified expression. This ensures that the expression as a whole appears on the same level of writing.

$$\gg S^{\tilde{x}}$$

This dot 5 begins the modified expression \tilde{x} (which is in the superscript position). It will not be misread as a baseline indicator because nothing comes between it and the superscript indicator.

$$\gg S_{\tilde{x}}$$

This dot 5 begins the modified expression \tilde{x} (which is in the subscript position). It will not be misread as a baseline indicator because nothing comes between it and the subscript indicator.

$$\gg \tilde{x}A_1$$

The first dot 5 begins the modified expression \tilde{x} (which is a left subscript to the letter A). The second dot 5 is a baseline indicator.

If the modified expression occurs in the middle or at the end of the superscript or subscript, the appropriate level indicator must be repeated before the multipurpose indicator to show continuation of the same level of writing. This ensures that the multipurpose indicator will not be misread as a baseline indicator.

$\gg S_{\tilde{x}+\tilde{y}}$

The first dot 5 begins the modified expression \tilde{x} . The second dot 5 begins the modified expression \tilde{y} . To ensure that the second dot 5 is not read as a baseline indicator, the subscript level is restated before the dot 5.

Since the multipurpose indicator is absent in the contracted form of "bar over" or "bar under", the level continues with certainty. It is not necessary to repeat the level indicator when the contracted form is used.

$\gg e^{a\bar{x}}$

PRACTICE 12F

More About Superscripts and Subscripts

(1) $a_0\bar{x}^n + a_1\bar{x}^{n-1}$

(2) $W = \frac{2}{3}\pi r^3 \underline{w} \left(h + \frac{3}{8}r \right)$

(3) $S^{\tilde{x}}$ and $S^{\tilde{x}+\tilde{y}}$

(4) $D_{\bar{x}}$ or $D_{\bar{x}+\bar{y}}$

(5) $3_{\bar{x}} - 2_{\bar{x}}$

(6) $\bar{n} A_1$

Modified Signs of Comparison

12.11 Definition

A modified sign of comparison consists of a simple sign of comparison, such as the equals sign or the tilde, modified by a caret, dot, triangle, question mark, vertical bar, or any symbol except another sign of comparison.

When a simple sign of comparison occurs above or below another simple sign of comparison the combination is transcribed as a sign of comparison compounded vertically. See Section 5.8 in Lesson 5 for a review of that construction. Note that many of those signs are printed with a single horizontal line "bar over" or "bar under". For example, "bar over greater than, inclusion with bar under, bar over single tilde, logical sum with bar under," etc. These signs are not to be misinterpreted as a horizontal bar modifying a sign of comparison.

12.12 Transcription


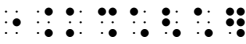


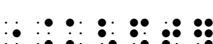


A modified sign of comparison is transcribed in accordance with the five-step rule for modified expressions.

In addition to the caret and inverted caret seen earlier in this lesson, you may also encounter a left- or right-pointing caret in a modified sign of comparison. Do not confuse these two symbols with the "less than" and "greater than" comparison signs. Ask an expert if context does not clarify the identity of this symbol.

⠠⠠⠠⠠⠠⠠⠠	Left-Pointing Caret	<
⠠⠠⠠⠠⠠⠠⠠	Right-Pointing Caret	>

The following list contains the modified equals signs most commonly used.

Modified Equals Sign		
⠠⠠⠠⠠⠠⠠⠠⠠	Caret Over Equals Sign	≐
⠠⠠⠠⠠⠠⠠⠠⠠	Caret Under Equals Sign	≑
	("is projective to")	
⠠⠠⠠⠠⠠⠠⠠⠠	Inverted Caret Over Equals Sign	≒
⠠⠠⠠⠠⠠⠠⠠⠠	Left-Pointing Caret Over Equals Sign	≓
⠠⠠⠠⠠⠠⠠⠠⠠	Right-Pointing Caret Over Equals Sign	≔

	Dot Over Equals Sign ("is approximately equal to")	\doteq
	Dot Over and Dot Under Equals Sign	$\dot{=}$
	Two Dots Over and Two Dots Under Equals Sign	$\ddot{=}$
	Hollow Dot Over Equals Sign ("is equal in degrees to")	$\overset{\circ}{=}$
	Equilateral Triangle Over Equals Sign	\triangleq
	Question Mark Over Equals Sign	$\stackrel{?}{=}$
	Vertical Bar Over Equals Sign	$\bar{=}$

➤ $\sqrt{5} \doteq 2.236$








➤ $5(3) + 2(-2) \stackrel{?}{=} 11$



If a symbol is encountered which does not appear on this list, construct a symbol in accordance with the same principles.

➤ $A \stackrel{m}{=} B$ 

The remainder of the list provided in the code contains other modified comparison signs most commonly used.

Modified Tilde		
	Dot Under Tilde	$\underset{\cdot}{\sim}$
	Dot Over Tilde	$\overset{\cdot}{\sim}$
Modified Horizontal Bar		
	Caret Over Bar	$\overset{\wedge}{-}$
	Caret Under Bar	$\underset{\wedge}{-}$
	("is perspective to")	
	Dot under Bar	$\underset{\cdot}{-}$

If the horizontal bar is modified by a dot *over* it, the combination is a modified sign of operation ("minus with dot over" signifying "proper difference"). The five-step rule is not used for this symbol. See Section 5.4.7 in Lesson 5.

Instructions: Use the five-step rule to show the horizontal grouping sign in the last item.

PRACTICE 12G

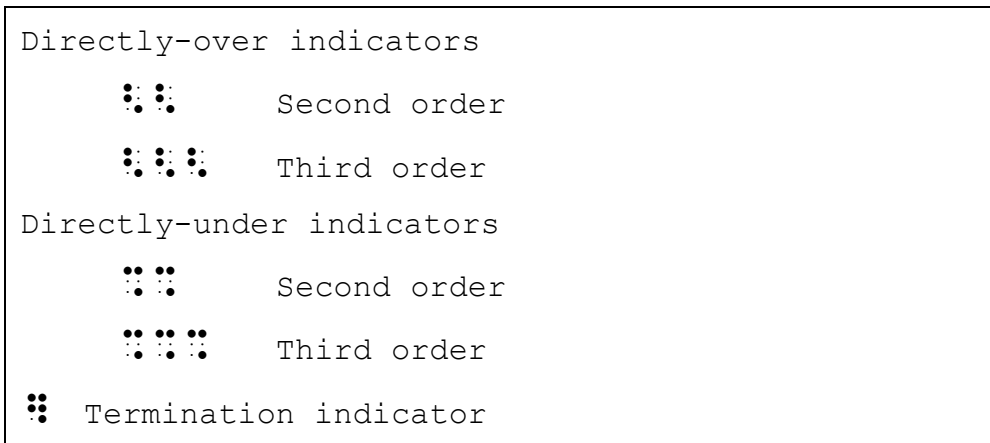
Modified Signs of Comparison and More

- $A \doteq 3.14r^2$
- $\angle b \doteq \frac{1}{2} \widehat{EB}$
- The symbol \triangleq is used to make a definition.
- $x \sim \mathcal{N}(0, 1)$
- $x^n = \underbrace{x \cdot x \cdot x \cdot \dots \cdot x}$

Expressions with More Than One Modifier

12.13 Modifiers of Higher Order

When two or more modifiers occur one above the other and apply to exactly the same expression, the second, third, etc. modifiers are "modifiers of higher order." In such cases, the directly-over indicator or the directly-under indicator is doubled, tripled, etc., before each modifier to indicate its position. The termination indicator is used only after the last modifier shown.



$$\ggg \frac{a=3}{x+y}$$



Analysis: The expression $x + y$ is modified directly over with a horizontal bar which, in turn, is modified directly over with the expression $a = 3$.

$$\ggg \frac{x+y}{\frac{a=3}{b=2}}$$



Analysis: The expression $x + y$ is modified directly under with a horizontal bar which, in turn, is modified directly under with the expression $a = 3$ which is modified directly under with the expression $b = 2$.

$$\gg \sum_{\substack{i, j, k \\ i < j < k}}$$



Analysis: The Sigma is modified directly under with letters i, j, k. That, in turn, is modified directly under with the expression $i < j < k$. Review the rules regarding use of the English-letter indicator in Lesson 3. An English-letter indicator is needed for the first letter "j" because it is a "single letter" according to the definition in the Nemeth Code. The other letters are either touching indicators or are preceded or followed by a sign of comparison, so an English-letter indicator is not needed.

12.13.1 **Parallel Horizontal Bars.** When two or more parallel horizontal bars are the same length and apply to exactly the same expression, they are treated as a single modifier. In such cases, the directly-over or the directly-under indicator is used only once, and the symbol for the bar is used as many times as necessary to correspond to the print text.

$$\gg \bar{9} \quad \dots \dots \dots \dots \dots$$

$$\gg (\bar{A}) \quad \dots \dots \dots \dots \dots$$

$$\gg \underset{|||}{x} \quad \dots \dots \dots \dots \dots$$

Attention: Two parallel horizontal bars must not be misinterpreted as the equals sign, and three parallel horizontal bars must not be misinterpreted as the identity sign. The symbol's meaning can be determined by reading the surrounding text for context. See Section 5.8 in Lesson 5 ("Signs of Comparison Compounded Vertically") to review equals sign printed above or below a sign of comparison.

12.14 Individual Modifiers

When two or more modifiers do not apply to exactly the same symbols but cover different portions of the same expression, the longer modification encloses the entire modified expression. Within the long expression, each inner expression is modified individually.

$$\gg \overline{\underline{x} \times \underline{x}} \quad \dots \dots \dots \dots \dots$$

Analysis: In the simulated braille below, the longer modification indicators and symbols are highlighted; the inner modified expressions are underlined.



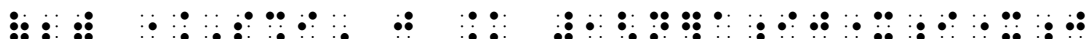
As you study the eight constructions in the [Example 12-19](#) note the use or nonuse of the English-letter indicator and the use or nonuse of the numeric indicator. The simulated braille is in Nemeth. Code switch indicators are omitted.

Example 12-19 |

(1) $\sum_{k=1}^n (2k) = n(n + 1)$



(2) $\sum_{i,j=1}^n a_{ij}x_i x_j$



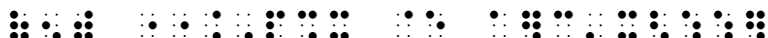
(3) $\sum_{x_1=0}^1 \sum_{x_2=0}^1$



(4) $\bigcup_{r=1}^n A_r = A_1 \cup A_2 \dots$



(5) $\overline{\prod_{x \in a} c'x}$



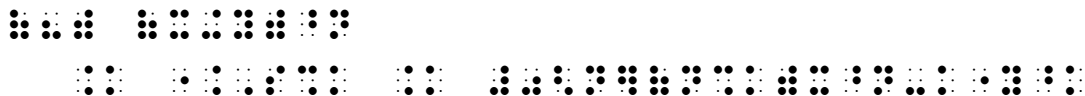
(6) $\frac{x+y}{a=3}^{b=2}$



(7) $\sum_{i=1}^6 \overline{P_{i-1}P_i}$



$$(8) \quad (x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$



This is a linked expression where $(x + y)^n$ is the anchor and

$= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ is the link. Recall from Lesson 8 that when a linked

expression will not fit on one braille line, start a runover line with the link.

The equals sign within the modification is not a suitable division site because the components of an expression modified according to the five-step rule should not be divided between lines. (See [Section 12.2.](#))

PRACTICE 12H

Expressions with More Than One Modifier

1. $\overline{\overline{9 \cdot 3}} = \overline{\overline{9 \cdot 3}}$

5. $\overline{x_a + y^n}$

2. $\overline{A^n}$

6. $\overline{\overline{N}}$

3. $\overline{A \cap B \cap C}$

7. $\sum_{i=1}^k \sum_{j=1}^k$

4. $\overline{a_n + b_p}$

8. $\prod_{\substack{j=1 \\ j \neq k}}^n$

Format

12.16 Formal Proof [NC Rule 26.7]

A proof is a valid argument that establishes the truth of a mathematical statement. It is often introduced by a heading such as *Theorem*, *Proposition*, or *Lemma*. A progressive sequence of statements leads to the conclusion. In a formal proof, every step of the argument is shown, and each step is supported by a definition or by a previously proven statement.

Lesson 11 explained how to transcribe a mathematical statement. Those guidelines are summarized in items a-c, below, in the context of a proof.

- a. A blank line precedes the beginning of the proof. (See the comment regarding line 1 in [Example 12-20](#) for an exception.)
- b. *Heading*: The heading can be formatted as a paragraph heading or as a cell-5 or cell-7 heading, at the transcriber's discretion. Review Section 11.38.b in Lesson 11 for details.
- c. *Statement*: Continue with the text, using normal (3-1) paragraph style. When the statement is printed in a variant typeform and the proof follows, in regular type, it is recommended that the typeform be preserved for the statement in order to retain distinction.
- d. *Auxiliary Captions*: Paragraph headings such as Given, Hypothesis, Prove, or Conclusion begin in cell 3, without a blank line before the paragraph. Associated material follows the caption. Runovers go in cell 1.
- e. *Two-Column Proof*: See [Section 12.16.1](#).
- f. *End of Proof Icon*: See [Section 12.16.2](#).
- g. When the proof is complete, insert a blank line before continuing with the text.

12.16.1 **Two-Column Proof**. When a formal proof is presented by numbered steps printed in two columns, the layout is changed as follows.

- a. If there is a caption such as "Proof", follow the same pattern established in [Section 12.16.d](#), "Auxiliary Captions".
- b. The column format is changed to a list in braille. A transcriber's note must call attention to the change in format. See [Section 12.16.1.c](#) for a sample transcriber's note.

A blank line is inserted before the list. If there are column headings, such as "Statement" and "Reason", see [Section 12.16.1.c](#). Each step begins in cell 1, starting with the first item from the left column. Runovers are in cell 3. The related item from the right column begins in cell 1 on the next line, with runovers in cell 3.

- c. *Identifiers*: Each item must be labeled with an identifier. Typically, the print copy includes the column headings "Statements" and "Reasons". In braille, the column headings are replaced with a letter – "S" or "R", respectively – as part of each item number.

Line 1: The proof may begin on line 1 providing lines 24 and 25 of the preceding page are blank (code switch indicators, running footers, and page numbers excepted); otherwise line 1 must be blank preceding the proof. Box lines are not required.

Line 1: The paragraph heading is printed in capital letters and also in a nonregular typeface (boldface). The capitals are retained and typeform (boldface) is disregarded.

Lines 1-2: Distinctive typeform (boldface) is retained in the statement.

Lines 3-6: Each auxiliary caption follows print regarding typeform (italics, in this example), and uses a 3-1 paragraph style.

Line 7: A blank line precedes the list.

Lines 8-15: Each item in the 2-column proof begins in cell 1, with runovers in cell 3. Identifying letters S and R are combined with each step number.

Line 15: A dark square is printed in the right margin to mark the end of the proof. The "qed" icon is transcribed.

Line 16: A blank line follows the proof.

PRACTICE 12I

Instructions: Create a Transcriber's Notes page that would appear in a volume which contains the proof shown in Practice 12J. The first paragraph states the title and edition of the Nemeth code as well as any BANA updates. *This statement is required in every volume that uses Nemeth.*

Mathematical content is transcribed according to *The Nemeth Braille Code for Mathematics and Science Notation 2022*.

The second paragraph explains the step-number format as described in [Section 12.16.1.c](#). Refer to *Braille Formats* for further guidelines regarding the structure of a Transcriber's Notes page.

PRACTICE 12J

Given: $3x = 7 - \frac{1}{2}x$

To Prove: $x = 2$

STEP	REASON
1. $3x = 7 - \frac{1}{2}x$	1. GIVEN
2. $6x = 14 - x$	2. Multiplication Property
3. $7x = 14$	3. Addition Property
4. $x = 2$	4. Division Property

For further practice, see Addendum 1—Reading Practice.

Submit Exercise 12 to your instructor.

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PRACTICE 12D

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PRACTICE 12E

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